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Some inverse problems with partial data
(Quelques problèmes inverses avec des données partielles)

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To the memory of Herman and Anna

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Contents

Thesis outline

The thesis consists of three parts dealing with three independent problems. All of those problems have in common that they are directly or indirectly related to inverse problems with partial data. Each of the parts has its own introduction, so here we only briefly outline their content. Concluding remarks for all of the parts are given in separate section at the end.

In Part 1, we consider partially overdetermined boundary-value problem for Laplace PDE in a planar simply connected domain with Lipschitz boundary $\partial\Omega$. Assuming Dirichlet and Neumann data available on $\Gamma \subset \partial\Omega$ to be real-valued functions in $W^{1/2,2}(\Gamma)$ and $L^2(\Gamma)$ classes, respectively, we develop a non-iterative method for solving this ill-posed Cauchy problem choosing the L^2 norm of the solution on $\partial\Omega\backslash\Gamma$ as a regularizing parameter. The present complex-analytic approach also naturally allows imposing additional pointwise constraints on the solution which, on the practical side, can help incorporating outlying boundary measurements without changing the boundary into a less regular one. Success of this work is based on a technical observation about explicit solvability of certain infinite-dimensional system of ODEs establishing a link between the approximation quality and regularization constraint. Such a link makes the regularizing scheme, which was used in different contexts before, into a non-iterative computational method. Part of the results of this work is to appear in the Journal of Inverse and Ill-Posed Problems (accepted February 2016).

Part 2 is concerned with the spectral structure of truncated Poisson operator. An eigenvalue problem for integral operator with Poisson kernel on a bounded domain is expected to produce an efficient basis for the representation of specific functions. Indeed, the structure of these eigenfunctions encodes harmonicity and geometry related to the problem whose solutions we seek to either interpolate from pointwise measurements or extrapolate beyond the measurement area. We study the one-dimensional version of this equation which turns out to be a long-standing problem. We establish interesting properties of solutions, discuss connections with other problems and develop original methods for the construction of asymptotic solution for large and small values of the geometric parameter. These asymptotic constructions stem from subtle analysis of structure of the problem yielding reductions to simpler integral equations (on a half-line) and second-order ODEs. Interestingly enough, integral equations with the same kernel appear in many different fields of physics: from electrostatics and viscous fluid motion to statistics of quantum gases and theory of stochastic processes. These particular instances of the equation have been subject to intense investigation over the last 60 years.

In Part 3, we deal with a particular inverse problem arising in a real physical experiment performed with SQUID microscope by our geophysics partners at the Paleomagnetism Lab in the Earth, Atmospheric and Planetary Sciences Department of Massachusetts Institute of Technology. The practical aim is to recover certain magnetization features (typically net moment, i.e. essentially an average magnetization) of a sample from partial measurements of one component of magnetic field above it. We pursue this goal by developing two new methods of solving this badly ill-posed problem. One of them is an adaptation, due to construction of Kelvin transformations, of tools from spherical geometry setting to the planar case considered. Another one is based on asymptotic analysis in Fourier domain where the matching is performed for the wave vectors of different magnitude. We advance in two directions. First of all, we perform constructive investigation of possibilities to extract the net moment (as well as other scalar quantities) from completely available data for either the scalar potential or for the normal component of the magnetic field. Second, we obtain practical formulas for computing net moment in the case of partially available data. In the first case, we provide a certain representation of the exact solution of the net moment problem, whereas in the second case, we construct asymptotic estimates based on the original idea of measurement extension. It is remarkable that recovery of tangential and normal components of the net moment require separate treatment based on different ideas.

PART 1

Recovery of harmonic functions from partial boundary data respecting internal pointwise values

1.1 Introduction

Many stationary physical problems are formulated in terms of reconstruction of a harmonic function in a planar domain from partially available measurements on its boundary. As it is often the case, the values of both the function and its normal derivative are available only on part of the boundary whereas the main interest is to determine the values inside the domain or on the unaccessible part of the boundary, or sometimes even the position of this complementary part of the boundary [3]. The planar formulation is a simplification that typically arises from original three-dimensional settings whose symmetry properties allow reformulation of the model in dimension two.

The Cauchy problem for Laplace's equation is known to be ill-posed: the famous Hadamard's example demonstrates the lack of continuous dependence of the solution on boundary data. This reveals the necessary compatibility between Dirichlet and Neumann data for the existence of a physically meaningful solution and advocates the use of regularization techniques.

Partially overdetermined problems for the elliptic operators have been vastly considered in various frameworks (see [25] and references therein) and different methods of their regularization and solution have been developed and investigated.

In the present work, we revisit the very classical setting - Laplace's PDE on a simply connected domain with Lipschitz boundary. Namely, we consider the prototypical case where the domain is the unit disk $\Omega = \mathbb{D}$, which is justified by the conformal invariance of the Laplace operator. We assume real-valuedness and appropriate regularity of the boundary data on a strict subset $\Gamma \subset \mathbb{T} := \partial\mathbb{D}$ required for the existence of a unique weak

$W^{1,2}(\Omega)$ solution:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = u_0, \quad \partial_n u = w_0 & \text{on} \quad \Gamma \quad \text{with} \quad u_0 \in W^{1/2,2}(\Gamma), \;\; w_0 \in L^2(\Gamma). \end{cases} \tag{1.1}$$

We employ a complex-analytic approach which has proven to be rather efficient in dealing with this [5, 7, 11, 12] and more general formulations of the problem: annular setting [27, 31], conductivity PDE [20] and their mixture [6].

Recall that if a function $g = u + iv$ is analytic (holomorphic), then u and v are real-valued harmonic functions satisfying the Cauchy-Riemann equations $\partial_n u = \partial_t v$, $\partial_t u = -\partial_n v$, where the partial derivatives are taken with respect to polar coordinates. Applied to problem (1.1), the first of these equations suggests that knowing w_0, one can, up to an additive constant, recover v on Γ, and therefore both u_0 and w_0 define the trace on Γ of the function g analytic inside Ω. However, the knowledge of an analytic function on a subset $\Gamma \subset \mathbb{T}$ of positive measure completely defines this function inside the whole domain (unit disk $\mathbb{D}$) [24, 37]. Of course, available data u_0, w_0 on Γ may not be compatible to yield the restriction of an analytic function onto Γ. This fact illustrates ill-posedness of the problem from the complex analysis point of view. At the same time, it leads to a natural regularization scheme that consists of finding a compatible set of data which is the closest to the original one and whose continuation behaves well on the unaccessible part of the boundary.

The described procedure can be formalized as a best norm-constrained approximation problem in Hardy space for the disk casted in the works [5, 7]. Pursueing this approach, we extend previously obtained results as follows.

First of all, we generalize the method in order to allow internal pointwise constraints on the solution. We rederive the solution formula and carry out analysis of the approximation quality for this case. One practical aspect of this modification might be a possibility to effectively process measurements from sensors positioned off the naturally smooth boundary by clustering these outlying measurements into a few points located inside the domain. We note that here internal pointwise data do make sense due to the analytical structure of the present framework - an advantage of working in Hardy rather than Lebesgue spaces. The possibility of imposing finite or infinite number of internal pointwise constraints on analytic function in the disk is classical [40] and has been studied from different viewpoints (e.g. [10]).

Second, we improve the previous solution algorithm which was an iterative procedure. As before, the solution formula is implicit for it contains a parameter to be chosen to satisfy the regularization constraint. However, if this adjustment previously had to be done by dichotomy, we now provide an expression allowing one to estimate this parameter directly from the regularization bound and thus avoid repetitive solution of the problem.

Lastly, we prove stability of the regularized problem with respect to all input data - a technical issue that appears not to have been raised before.

This Part is organized as follows. Section 1.2 provides an introduction to the theory of Hardy spaces which are essential functional spaces in the present approach. In Section 1.3, we formulate the problem, prove existence of a unique solution and give its useful characterization. Section 1.4 discusses the choice of interpolation function which is a technical tool to prescribe desired values inside the domain; we also provide an alternative form of the solution that turns out to be useful later. In Section 1.5, we obtain specific balance relations governing approximation rate on a given subset of the circle and discrepancy on its complement, which shed light on the quality of the solution depending on a choice of some auxiliary parameters. Also, at this point we introduce a novel series expansion method of evaluation of quantities governing solution quality. Section 1.6 introduces a closely related problem whose solution might be computationally cheaper in certain cases. We further look into sensitivity of the solution to perturbations of all input data in Section 1.7, addressing the stability issue and providing technical estimates. We conclude with Section 1.8 by presenting numerical illustrations of certain properties of the solution, a short discussion of the choice of technical parameters, and suggestion of a new efficient computational algorithm based on the results in Section 1.5.

1.2 Background in the theory of Hardy spaces

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ with boundary $\mathbb{T}$.

Hardy spaces $H^p(\mathbb{D})$ can be defined as classes of holomorphic functions on the disk with finite norms

$$\|F\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^p d\theta \right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|F\|_{H^\infty} = \sup_{|z|<1} |F(z)| .$$

These are Banach spaces that enjoy plenty of interesting properties, and they have been studied in detail over the years [18, 23, 26, 40]. In this section we give a brief introduction into the topic, yet trying to be as much self-contained as possible, adapting general material to our particular needs.

The key property of functions in Hardy spaces is their behavior on the boundary $\mathbb{T}$ of the disk. More precisely, boundary values of functions belonging to the Hardy space H^p are well-defined in the L^p sense

$$\lim_{r \nearrow 1} \|F(r\cdot) - F(\cdot)\|_{L^p(\mathbb{T})} = 0, \quad 1 \le p < \infty, \tag{1.2}$$

as well as pointwise, for almost every $\theta \in [0, 2\pi]$:

$$\lim_{r \nearrow 1} F\left(re^{i\theta}\right) = F\left(e^{i\theta}\right). \tag{1.3}$$

It is the content of Fatou's theorem (see, for instance, [26]) that the latter limit exists almost everywhere not only radially but also along any non-tangential path. Thanks to Parseval's identity, the proof of (1.2) is especially simple when $p = 2$ (see [32, Thm 1.1.10]), the case that we will work with presently.

Given a boundary function $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$ whose Fourier coefficients of negative index vanish

$$f_{-n} := \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) e^{in\theta} d\theta = 0, \quad n = 1, 2, \ldots, \tag{1.4}$$

(in this case, we say $f \in H^p(\mathbb{T})$), there exists $F \in H^p(\mathbb{D})$ such that $F\left(re^{i\theta}\right) \to f\left(e^{i\theta}\right)$ in L^p as $r \nearrow 1$, and it is defined by the Poisson representation formula, for $re^{i\theta} \in \mathbb{D}$,

$$F\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) P_r(\theta - t)\, dt, \tag{1.5}$$

where we employed the Poisson kernel for $\mathbb{D}$

$$P_r(\theta) := \frac{1-r^2}{1-2r\cos\theta + r^2} = \sum_{k=-\infty}^{\infty} r^{|k|}e^{ik\theta}\,,\ 0<r<1\,,\quad \theta\in[0,2\pi]\,.$$

Note that the vanishing condition for the Fourier coefficients of negative order is equivalent to the requirement of the Poisson integral (1.5) to be analytic in $\mathbb{D}$. Indeed, since $f\left(e^{i\theta}\right) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}$, the right-hand side of (1.5) reads

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) P_r(\theta-t)\, dt &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} r^{|k|}e^{ik\theta}\sum_{n=-\infty}^{\infty} f_n \int_0^{2\pi} e^{i(n-k)t}dt = \sum_{n=-\infty}^{\infty} f_n r^{|n|}e^{in\theta}\\ &= f_0 + \sum_{n=1}^{\infty}\left(f_n z^n + f_{-n}\bar{z}^n\right),\end{aligned}$$

and hence, if we want this to define a holomorphic function through (1.5), we have to impose condition (1.4).

Because of the established isomorphism, we can identify the space $H^p = H^p(\mathbb{D})$ with $H^p(\mathbb{T}) \subset L^p(\mathbb{T})$ for $p \geq 1$ (the case $p=1$ requires more sophisticated reasoning invoking F. & M. Riesz theorem [26]). It follows that H^p is a Banach space (as a closed subspace of $L^p(\mathbb{T})$ which is complete), and we have inclusions due to properties of Lebesgue spaces on bounded domains

$$H^\infty \subseteq H^s \subseteq H^p, \quad s \geq p \geq 1. \tag{1.6}$$

Summing up, we can abuse notation by employing only one letter f, and write

$$\|f\|_{H^p} = \|f\|_{L^p(\mathbb{T})} \tag{1.7}$$

whenever $f \in L^p(\mathbb{T})$, $p \geq 1$, satisfies (1.4).

Moreover, in case $p = 2$, which we will focus on, Parseval's identity provides an isometry between the Hardy space $H^2 = H^2(\mathbb{D})$ and the space $l_2(\mathbb{N}_0)$ of square-summable sequences [1]. Hence, H^2 is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\overline{g\left(e^{i\theta}\right)}\, d\theta = \sum_{k=0}^{\infty} f_k \bar{g}_k. \tag{1.8}$$

We will also repeatedly make use of the fact that H^∞ functions act as multipliers in H^p, that is, $H^\infty \cdot H^p \subset H^p$. There is another useful property of Hardy classes to perform factorization: if $f \in H^p$ and $f(z_j) = 0$, $z_j \in \mathbb{D}$,

[1]Here and onwards, we stick to the convention: $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, $\mathbb{N}_+ := \{1, 2, 3, \dots\}$.

$j = 1, \ldots, N$, then $f = bg$ with $g \in H^p$ and the finite Blaschke product $b \in H^\infty$ defined as

$$b(z) = e^{i\phi_0} \prod_{j=1}^{N} \left(\frac{z - z_j}{1 - \bar{z}_j z} \right) \tag{1.9}$$

for some constant $\phi_0 \in [0, 2\pi]$. Possibility of such factorization comes from the observation that each factor of $b(z)$ is analytic in $\mathbb{D}$ and automorphic since

$$|z|^2 + |z_j|^2 - |z|^2 |z_j|^2 = |z|^2 \left(1 - |z_j|^2 /2\right) + |z_j|^2 \left(1 - |z|^2 /2\right) \leq 1,$$

and thus

$$\left| \frac{z - z_j}{1 - \bar{z}_j z} \right|^2 = \frac{1 - 2\mathrm{Re}\,(\bar{z}_j z) + |z|^2 + |z_j|^2 - 1}{1 - 2\mathrm{Re}\,(\bar{z}_j z) + |z|^2 |z_j|^2} \leq 1.$$

Additionally, this shows that

$$|b| \equiv 1, \quad z \in \mathbb{T}, \tag{1.10}$$

and hence $\|b\|_{H^\infty} = 1$.

We let $\bar{H}_0^2$ denote the orthogonal complement of H^2 in $L^2(\mathbb{T})$, so that $L^2 = H^2 \oplus \bar{H}_0^2$. Recalling the characterization (1.4) of H^2 functions, we can view $\bar{H}_0^2$ as the space of functions whose expansions have non-vanishing Fourier coefficients of only negative index, and hence it characterizes $L^2(\mathbb{T})$ functions which are holomorphic in $\mathbb{C} \backslash \bar{\mathbb{D}}$ and decay to zero at infinity.

Similarly, we can introduce the orthogonal complement to bH^2 in $L^2(\mathbb{T})$ with b as in (1.9) so that $L^2 = bH^2 \oplus \left(bH^2\right)^\perp$ which in its turn decomposes into a direct sum as $\left(bH^2\right)^\perp = \bar{H}_0^2 \oplus \left(bH^2\right)^{\perp_{H^2}}$ with $\left(bH^2\right)^{\perp_{H^2}} \subset H^2$ denoting the orthogonal complement to bH^2 in H^2; it is not empty if $b \not\equiv \mathrm{const}$, whence the proper inclusion $bH^2 \subset H^2$ holds. Moreover, making use of the Cauchy integral formula, it can be shown that

$$\left(bH^2\right)^{\perp_{H^2}} := \left(bH^2\right)^\perp \ominus \bar{H}_0^2 = \frac{P_{N-1}(z)}{\prod_{j=1}^{N} (1 - \bar{z}_j z)},$$

where $P_{N-1}(z)$ is the space of polynomials of degree at most $N-1$ in z.

An operator A is called a Toeplitz operator on H^2 if its matrix in the Fourier basis has constant elements along all diagonals: $A_{k,m} := \langle Az^k, z^m \rangle_{L^2(\mathbb{T})}$ depends only on the difference $|k - m|$ for $k, m = 0, 1, 2, \ldots$.

We will need a spectral result on Toeplitz operators known as Hartman-Wintner theorem. Its proof can be found in [17, 35] and also, in a self-consistent manner, in Appendix.

Given $J \subset \mathbb{T}$, let us introduce the Toeplitz operator ϕ with symbol χ_J (the indicator function of J), defined by:

$$\begin{aligned} H^2 &\rightarrow H^2 \\ F &\mapsto \phi(F) = P_+(\chi_J F), \end{aligned} \tag{1.11}$$

where we let P_+ denote the orthogonal projection from $L^2(\mathbb{T})$ onto H^2 (that might be realized by setting Fourier coefficients of negative index to zero or convolving the function with the Cauchy kernel). Similarly, $P_- := I - P_+$ defines the orthogonal projection onto $\bar{H}_0^2$.
We also notice that the map $L^2(\mathbb{T}) \to bH^2 : F \mapsto bP_+\left(\bar{b}F\right)$ is the orthogonal projection onto bH^2. Indeed, taking into account (1.10), for any $u \in L^2(\mathbb{T})$, $v \in H^2$,

$$\left\langle u - bP_+\left(\bar{b}u\right), bv\right\rangle_{L^2(\mathbb{T})} = \langle u, bv\rangle_{L^2(\mathbb{T})} - \left\langle P_+\left(\bar{b}u\right), \bar{b}bv\right\rangle_{L^2(\mathbb{T})} = 0.$$

Any function in H^p, $p \geq 1$, being analytic and sufficiently regular on $\mathbb{T}$, admits integral representation in terms of its boundary values and thus is uniquely determined by means of the Cauchy formula. However, it is also possible to recover a function f holomorphic in $\mathbb{D}$ from its values on a subset of the boundary $I \subset \mathbb{T}$ using so-called Carleman's formulas [4, 24]. Write $\mathbb{T} = I \cup J$ with I and J being Lebesgue measurable sets.

Proposition 1.2.1. *Assume $|I| > 0$ and let $\Phi \in H^\infty$ be any function such that $|\Phi| > 1$ in $\mathbb{D}$ and $|\Phi| = 1$ on J. Then, $f \in H^p$, $p \geq 1$ can be represented from $f|_I$ as*

$$f(z) = \frac{1}{2\pi i} \lim_{\alpha \to \infty} \int_I \frac{f(\xi)}{\xi - z} \left[\frac{\Phi(\xi)}{\Phi(z)}\right]^\alpha d\xi, \tag{1.12}$$

where the convergence is uniform on compact subsets of $\mathbb{D}$.

Proof. Since $\Phi \in H^\infty$ and $f \in H^p \subseteq H^1$, it is clear that $f(z)[\Phi(z)]^\alpha \in H^1$, and so the Cauchy formula applies to $f(z)[\Phi(z)]^\alpha = f(z)\exp[\alpha \log \Phi(z)]$ for any $\alpha > 0$

$$f(z)[\Phi(z)]^\alpha = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\xi)[\Phi(\xi)]^\alpha}{\xi - z} d\xi$$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i}\left(\int_I + \int_J\right) \frac{f(\xi)}{\xi - z}\left[\frac{\Phi(\xi)}{\Phi(z)}\right]^\alpha d\xi.$$

Since the second integral vanishes in absolute value as $\alpha \nearrow \infty$ for any $z \in \mathbb{D}$ (by the choice of Φ), we have (1.12). □

The integral representation (1.12) implies the following uniqueness result (see also e.g. [40, Thm 17.18], for a different argument based on the factorization which shows that $\log|f| \in L^1(\mathbb{T})$ whenever $f \in H^p$).

Corollary 1.2.1. *Functions in H^1 are uniquely determined by their boundary values on $I \subset \mathbb{T}$ provided that $|I| > 0$.*

It follows that if two H^p functions agree on a subset of $\mathbb{T}$ with non-zero Lebesgue measure, then they must coincide everywhere in $\mathbb{D}$. This complements the identity theorem for holomorphic functions [1] claiming that zero set of an analytic function cannot have an accumulation point inside the domain of analyticity which particularly implies that two functions coinciding in a neighbourhood of a point of analyticity are necessarily equal in the whole domain of analyticity.

Remark 1.2.1. *Using the isometry $H^2 \to \bar{H}_0^2$:*

$$f(z) \mapsto \frac{1}{z}\overline{f\left(\frac{1}{\bar{z}}\right)}, \; z \in \mathbb{D}$$

(which is clear from the Fourier expansion on the boundary), we check that Proposition 1.2.1 and Corollary 1.2.1 also apply to functions in $\bar{H}_0^2$.

Remark 1.2.2. *The auxiliary function Φ termed as "quenching" function can be chosen as follows. Let u be a Poisson integral of a positive function vanishing on J (for instance, the characteristic function χ_I) and v its harmonic conjugate that can be recovered (up to an additive constant) at $z = re^{i\theta}$, $r < 1$ by convolving u on $\mathbb{T}$ (using normalized Lebesgue measure $d\sigma = \frac{1}{2\pi}d\theta$) with the conjugate Poisson kernel Im$\left(\frac{1+re^{it}}{1-re^{it}}\right)$, $t \in [0, 2\pi]$, see [26] for details. Then, clearly, $\Phi = \exp(u + iv)$ is analytic in $\mathbb{D}$ and satisfies the required conditions. More precisely, combining the recovered v with the Poisson representation formula for u, we conclude that convolution of boundary values of u with the Schwarz kernel $\frac{1+re^{it}}{1-re^{it}}$, $t \in [0, 2\pi]$ defines (up to an additive constant) the analytic function $u(z) + iv(z)$ for $z = re^{i\theta} \in \mathbb{D}$. An explicit quenching function constructed in such a way will be given in Section 1.3 by (1.42).*

Remark 1.2.3. *A similar result was also obtained and discussed in [37], see also [4, 9, 28].*

As a consequence of Remark 1.2.1, we derive a useful tool in form of

Proposition 1.2.2. *The Toeplitz operator ϕ is an injection on H^2.*

Proof. By the orthogonal decomposition $L^2 = H^2 \oplus \bar{H}_0^2$, we have $\chi_J g = P_+(\chi_J g) + P_-(\chi_J g)$. Now, if $P_+(\chi_J g) = 0$, then $\chi_J g$ is a $\bar{H}_0^2$ function vanishing on I and hence, by Remark 1.2.1, must be identically zero. □

The last result for Hardy spaces that we are going to employ is the density of traces [7, 9].

Proposition 1.2.3. *Let $J \subset \mathbb{T}$ be a subset of non-full measure, that is $|I| = |\mathbb{T} \backslash J| > 0$. Then, the restriction $H^p|_J := (trH^p)|_J$ is dense in $L^p(J)$, $1 \le p < \infty$.*

Proof. In the particular case $p = 2$ (other values of p are treated in [7]), we prove the claim by contradiction. Assume that there is non-zero $f \in L^2(J)$ orthogonal to $H^2|_J$, then, extending it by zero on I, we denote the extended function as $\tilde{f}$. We thus have $\left\langle \tilde{f}, g\right\rangle_{L^2(\mathbb{T})} = 0$ for all $g \in H^2$ which implies $\tilde{f} \in \bar{H}_0^2$ and hence, by Remark 1.2.1, $f \equiv 0$.

□

Remark 1.2.4. *From the proof and Remark 1.2.1, we see that the same density result holds if one replaces H^2 with $\bar{H}_0^2$.*

There is a counterpart of Propositon 1.2.3 that also characterizes boundary traces of H^p spaces.

Proposition 1.2.4. *Assume $|I| > 0$, $f \in L^p(I)$, $1 \le p \le \infty$.* Let $\{g_n\}_{n=1}^\infty$ be a sequence of H^p functions such that $\lim_{n\to\infty} \|f - g_n\|_{L^p(I)} = 0$. Then, $\|g_n\|_{L^p(J)} \to \infty$ as $n \to \infty$ unless f is the trace of a H^p function.

Proof. Consider the case $1 < p < \infty$; for the cases $p = 1$ and $p = \infty$ we refer to [7] and [9], respectively. We argue by contradiction: assume that f is not the trace on I of some H^p function, but $\lim_{n\to\infty} \|g_n\|_{L^p(J)} < \infty$. Then, by hypothesis, the sequence $\{g_n\}_{n=1}^{\infty}$ is bounded not only in $L^p(J)$ but also in H^p. Since H^p is reflexive (as any $L^p(\mathbb{T})$ is for $1 < p < \infty$), it follows from the Banach-Alaoglu theorem (or see [30, Ch. 10 Thm 7]) that the closed unit ball in H^p is weakly compact, therefore, we can extract a subsequence $\{g_{n_k}\}$ that converges weakly in H^p: $g_{n_k} \rightharpoonup g$ for some $g \in H^p$. However, since $g_n \to f$ in $L^p(I)$, we must have $f = g|_I$, a contradiction. □

Remark 1.2.5. *When $|J| = 0$, the existence of a H^p sequence $\{g_n\}_{n=1}^{\infty}$ approximating $f \in L^p(I)$ in $L^p(I)$ norm, means that f actually belongs to H^p (which is a closed subspace of $L^p(\mathbb{T}) = L^p(I)$).*

1.3 An extremal problem and its solution

We consider the problem of finding a H^2 function which takes prescribed values $\{\omega_j\}_{j=1}^N \in \mathbb{C}$ at interior points $\{z_j\}_{j=1}^N \in \mathbb{D}$ which best approximates a given $L^2(I)$ function on a subset of the boundary $I \subset \mathbb{T}$ while remaining close enough to another $L^2(J)$ function on the complementary part $J \subset \mathbb{T}$.

We proceed with a technical formulation of this problem. Assuming given interpolation values at distinct interior points $\{z_j\}_{j=1}^N \in \mathbb{D}$, we let $\psi \in H^2$ be some fixed function satisfying the interpolation conditions

$$\psi(z_j) = \omega_j \in \mathbb{C}, \quad j = 1, \dots, N. \tag{1.13}$$

Then, any interpolating function in H^2 fulfiling these conditions can be written as $\tilde{g} = \psi + bg$ for arbitrary $g \in H^2$ with $b \in H^\infty$ the finite Blaschke product defined in (1.9).

As before, let $\mathbb{T} = I \cup J$ with both I and J being of non-zero Lebesgue measure. For the sake of simplicity, we write $f = f|_I \vee f|_J$ to mean a function defined on the whole $\mathbb{T}$ through its values given on I and J.

For $h \in L^2(J)$, $M \geq 0$, let us introduce the following functional spaces

$$\mathcal{A}^{\psi,b} := \left\{\tilde{g} \in H^2 : \tilde{g} = \psi + bg,\, g \in H^2\right\}, \tag{1.14}$$

$$\mathcal{B}_{M,h}^{\psi,b} := \left\{g \in H^2 : \|\psi + bg - h\|_{L^2(J)} \leq M\right\}, \tag{1.15}$$

$$\mathcal{C}_{M,h}^{\psi,b} := \left\{f \in L^2(I) : f = \psi|_I + b\, g|_I,\, g \in \mathcal{B}_{M,h}^{\psi,b}\right\}. \tag{1.16}$$

We then have inclusions $\mathcal{C}_{M,h}^{\psi,b} \subseteq \mathcal{A}^{\psi,b}\big|_I \subseteq H^2\big|_I \subset L^2(I)$ and $\mathcal{C}_{M,h}^{\psi,b} = \left(\psi + b\mathcal{B}_{M,h}^{\psi,b}\right)\Big|_I \neq \emptyset$ since $\mathcal{B}_{M,h}^{\psi,b} \neq \emptyset$ for any given $h \in L^2(J)$ and $M > 0$ as follows from Proposition 1.2.3.

Now the framework is set to allow us to pose the problem in precise terms.

Given $f \in L^2(I)$, our goal will be to find a solution to the following bounded extremal problem

$$\min_{g \in \mathcal{B}_{M,h}^{\psi,b}} \|\psi + bg - f\|_{L^2(I)}. \tag{1.17}$$

As it was briefly mentioned at the beginning, the motivation for such a formulation is to look for

$$\tilde{g}_0 := \psi + bg_0 \in \mathcal{A}^{\psi,b} \quad \text{such that} \quad g_0 = \arg \min_{g \in \mathcal{B}_{M,h}^{\psi,b}} \|\underbrace{\psi + bg}_{=\tilde{g}} - f\|_{L^2(I)}, \tag{1.18}$$

i.e. the best H^2-approximant to f on I which fulfils interpolation conditions (1.13) and is not too far from the reference h on J: $\|\tilde{g}_0 - h\|_{L^2(J)} \leq M$. In view of Proposition 1.2.4, the L^2-constraint on J is crucial whenever $f \notin \mathcal{A}^{\psi,b}\big|_I$ (which is always the case when known data are recovered from physical measurements necessarily subject to noise). In other words, we assume that

$$g|_I \neq \bar{b}(f - \psi), \tag{1.19}$$

i.e. there is no $\tilde{g} = \psi + bg \in H^2$ whose trace on I is exactly the given function $f \in L^2(I)$, and at the same time remains within the L^2-distance M from h on J. This motivates the choice (1.15) for the space of admissible solutions $\mathcal{B}_{M,h}^{\psi,b}$.

Existence and uniqueness of solution to (1.17) can be reduced to what has been proved in a general setting in [7]. Here we present a slightly different proof.

Theorem 1.3.1. *For any $f \in L^2(I)$, $h \in L^2(J)$, $\psi \in H^2$, $M \geq 0$ and $b \in H^\infty$ defined as (1.9), there exists a unique solution to the bounded extremal problem (1.17).*

Proof. By the existence of a best approximation projection onto a non-empty closed convex subset of a Hilbert space (see, for instance, [16, Thm 3.10.2]), it is required to show that the space of restrictions $\left.\mathcal{B}_{M,h}^{\psi,b}\right|_I$ is a closed convex subset of $L^2(I)$. Convexity is a direct consequence of the triangle inequality:

$$\left\|\alpha\left(bg_1 + \psi - h\right) + (1-\alpha)\left(bg_2 + \psi - h\right)\right\|_{L^2(J)} \leq \alpha M + (1-\alpha) M = M$$

for any $g_1,\ g_2 \in \mathcal{B}_{M,h}^{\psi,b}$ and $\alpha \in [0,1]$.

We will now show the closedness property. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}_{M,h}^{\psi,b}$ functions which converges in $L^2(I)$ to some g: $\|g - g_n\|_{L^2(I)} \to 0$ as $n \to \infty$. We need to prove that $g \in \mathcal{B}_{M,h}^{\psi,b}$.

We note that $g \in \left.H^2\right|_I$, since otherwise, by Proposition 1.2.4, $\|g_n\|_{L^2(J)} \to \infty$ as $n \to \infty$, which would contradict the fact that $g_n \in \mathcal{B}_{M,h}^{\psi,b}$ starting with some n. Therefore, $\psi + bg \in H^2$ and $\langle \psi + bg, \xi\rangle_{L^2(\mathbb{T})} = 0$ for any $\xi \in \bar{H}_0^2$, which implies that

$$\langle \psi + bg, \xi\rangle_{L^2(I)} = \langle (\psi + bg) \vee 0, \xi\rangle_{L^2(\mathbb{T})} = -\left\langle 0 \vee (\psi + bg), \xi\right\rangle_{L^2(\mathbb{T})} = -\langle \psi + bg, \xi\rangle_{L^2(J)}.$$

From here, using the same identity for $\psi + bg_n$, we obtain

$$\begin{aligned}\langle \psi + bg - h, \xi\rangle_{L^2(J)} &= -\langle \psi + bg, \xi\rangle_{L^2(I)} - \langle h, \xi\rangle_{L^2(J)} = -\lim_{n\to\infty} \langle \psi + bg_n, \xi\rangle_{L^2(I)} - \langle h, \xi\rangle_{L^2(J)} \\ &= \lim_{n\to\infty} \langle \psi + bg_n, \xi\rangle_{L^2(J)} - \langle h, \xi\rangle_{L^2(J)}.\end{aligned}$$

Since $g_n \in \mathcal{B}_{M,h}^{\psi,b}$ for all n, the Cauchy-Schwarz inequality gives

$$\left|\langle \psi + bg - h, \xi\rangle_{L^2(J)}\right| = \lim_{n\to\infty} \left|\langle \psi + bg_n - h, \xi\rangle_{L^2(J)}\right| \leq M \|\xi\|_{L^2(J)}$$

for any $\xi \in \left.\bar{H}_0^2\right|_J$. The final result is now furnished by employing density of $\left.\bar{H}_0^2\right|_J$ in $L^2(J)$ (Proposition 1.2.3 and Remark 1.2.4) and the dual characterization of $L^2(J)$ norm:

$$\|\psi + bg - h\|_{L^2(J)} = \sup_{\substack{\xi \in L^2(J)\\ \|\xi\|_{L^2(J)} \leq 1}} \left|\langle \psi + bg - h, \xi\rangle_{L^2(J)}\right| = \sup_{\substack{\xi \in \bar{H}_0^2\\ \|\xi\|_{L^2(J)} \leq 1}} \left|\langle \psi + bg - h, \xi\rangle_{L^2(J)}\right| \leq M.$$

□

A key property of the solution is that the constraint in (1.15) is necessarily saturated unless $f \in \mathcal{A}^{\psi,b}\big|_I$.

Lemma 1.3.1. *If $f \notin \mathcal{A}^{\psi,b}\big|_I$ and $g \in \mathcal{B}^{\psi,b}_{M,h}$ solves (1.17), then $\|\psi + bg - h\|_{L^2(J)} = M$.*

Proof. To show this, suppose the opposite, i.e. there is $g_0 \in H^2$ solving (1.17) for which we have

$$\|\psi + bg_0 - h\|_{L^2(J)} < M.$$

The last condition means that g_0 is in interior of $\mathcal{B}^{\psi,b}_{M,h}$, and hence we can define $g^\star := g_0 + \epsilon\delta_g \in \mathcal{B}^{\psi,b}_{M,h}$ for sufficiently small $\epsilon > 0$ and $\delta_g \in H^2$, $\|\delta_g\|_{H^2} = 1$ such that $\mathrm{Re}\,\langle b\delta_g, \psi + bg_0 - f\rangle_{L^2(I)} < 0$, where the equality case is eliminated by (1.19). By the smallness of ϵ, the quadratic term is negligible, and thus we have

$$\begin{aligned}\|\psi + bg^\star - f\|^2_{L^2(I)} &= \|\psi + bg_0 - f\|^2_{L^2(I)} + 2\epsilon\mathrm{Re}\,\langle b\delta_g, \psi + bg_0 - f\rangle_{L^2(I)} + \epsilon^2\,\|\delta_g\|^2_{L^2(I)} \\ &< \|\psi + bg_0 - f\|^2_{L^2(I)},\end{aligned}$$

which contradicts the minimality of g_0. □

As an immediate consequence of saturation of the constraint, we obtain

Corollary 1.3.1. *The requirement $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$ implies that the formulation of the problem should be restricted to the case $M > 0$.*

Proof. If $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$ and $M = 0$, the Lemma entails that $h \in \mathcal{A}^{\psi,b}\big|_J$. Then, $h = \psi + bg$ for some $g \in H^2$ and its extension to the whole $\mathbb{D}$ (given, for instance, by Proposition 1.2.1) uniquely determines $\tilde{g} = h$ without resorting to solution of the bounded extremal problem (1.17), hence independently of f. □

Having established that equality holds in (1.15), we approach (1.17) as a constrained optimization problem following a standard idea of Lagrange multipliers (e.g. [43]) and claim that for a solution g to (1.17) and for some $\lambda \in \mathbb{R}$, we must necessarily have

$$\langle \delta_{\tilde{g}}, (\tilde{g} - f) \vee \lambda(\tilde{g} - h)\rangle_{L^2(\mathbb{T})} = 0 \tag{1.20}$$

for any $\delta_{\tilde{g}} \in bH^2$ (recall that $\tilde{g} = \psi + bg$ and $\delta_{\tilde{g}} = b\delta_g$ for $\delta_g \in H^2$) which is a condition of tangency of level lines of the minimizing objective functional and the constraint functional. The condition (1.20) can be shown by the same variational argument as in the proof of Lemma 1.3.1, it must hold true, otherwise we would be able to improve the minimum while still remaining in the admissible set. This motivates us to search for $g \in H^2$ such that, for $\lambda \in \mathbb{R}$,

$$[(\psi + bg - f) \vee \lambda(\psi + bg - h)] \in \left(bH^2\right)^\perp \tag{1.21}$$

which is equivalent to

$$P_+\left[\bar{b}(\psi + bg - f) \vee \lambda\bar{b}(\psi + bg - h)\right] = 0. \tag{1.22}$$

Theorem 1.3.2. *If $f \notin \mathcal{A}^{\psi,b}\big|_I$, the solution to the bounded extremal problem (1.17) is given by*

$$g_0 = (1+\mu\phi)^{-1} P_+ \left(\bar{b}(f-\psi) \vee (1+\mu)\,\bar{b}(h-\psi)\right), \tag{1.23}$$

where the parameter $\mu > -1$ is uniquely chosen such that $\|\psi + bg_0 - h\|_{L^2(J)} = M$.

The proof of Theorem 1.3.2 goes in three steps.

1.3.1 Solution for the case $h = 0$

For simplicity, we first assume $h = 0$. Then, the equation (1.22) can be elaborated as follows

$$P_+ \left(\bar{b}(\psi + bg)\right) + (\lambda - 1) P_+ \left(0 \vee \bar{b}(\psi + bg)\right) = P_+ \left(\bar{b}f \vee 0\right),$$

$$g + P_+ \left(\bar{b}\psi\right) + (\lambda - 1) P_+ \left(0 \vee \bar{b}\psi\right) + (\lambda - 1)\,\phi g = P_+ \left(\bar{b}f \vee 0\right),$$

$$(1+\mu\phi)\, g = -P_+ \left(\bar{b}(\psi - f) \vee (1+\mu)\,\bar{b}\psi\right), \tag{1.24}$$

where we introduced the parameter $\mu := \lambda - 1 \in \mathbb{R}$.

The Toeplitz operator ϕ, defined as (1.11), is self-adjoint and, as it can be shown (see the Hartman-Wintner theorem in Appendix), its spectrum is

$$\sigma(\phi) = [\text{ess inf}\ \chi_J, \text{ess sup}\ \chi_J] = [0,1], \tag{1.25}$$

hence $\|\phi\| \le 1$ and the operator $(1+\mu\phi)$ is invertible on H^2 for $\mu > -1$ allowing to claim that

$$g = -(1+\mu\phi)^{-1} P_+ \left(\bar{b}(\psi - f) \vee (1+\mu)\,\bar{b}\psi\right). \tag{1.26}$$

This generalizes the result of [5] to the case when solution needs to meet pointwise interpolation conditions.

1.3.2 Solution for the case $h \neq 0$, $h \in H^2\big|_J$

Now, let $h \neq 0$, but assume it to be the restriction to J of some H^2 function.

We write $f = \varrho + \kappa|_I$ for $\kappa \in H^2$ such that $\kappa|_J = h$. Then, the solution to (1.17) is

$$g_0 = \arg \min_{g \in \mathcal{B}^{\psi,b}_{M,h}} \|\psi + bg - f\|_{L^2(I)} = \arg \min_{g \in \tilde{\mathcal{B}}_{M,0}} \left\|\tilde{\psi} + bg - \varrho\right\|_{L^2(I)},$$

where $\tilde{\psi} := \psi - \kappa$ and

$$\tilde{\mathcal{B}}_{M,0} := \left\{ g \in H^2 : \left\|\tilde{\psi} + bg\right\|_{L^2(J)} \le M \right\}.$$

It is easy to see that, due to $\kappa|_J = h$, we have $\tilde{\mathcal{B}}_{M,0} = \mathcal{B}^{\psi,b}_{M,h}$. Therefore, the already obtained results (1.24), (1.26) apply to yield

$$\begin{aligned}(1+\mu\phi)\, g_0 &= -P_+\left(\bar{b}\left(\tilde{\psi}-\varrho\right)\vee(1+\mu)\,\bar{b}\tilde{\psi}\right)\\ &= -P_+\left(\bar{b}\,(\psi-\kappa-\varrho)\vee(1+\mu)\,\bar{b}\,(\psi-\kappa)\right)\\ &= P_+\left(\bar{b}\,(f-\psi)\vee(1+\mu)\,\bar{b}\,(h-\psi)\right), \qquad (1.27)\end{aligned}$$

from where (1.23) follows.

1.3.3 Solution for the case $h \neq 0,\ h \in L^2(J)$

Here we assume $h \notin H^2\big|_J$ but only $h \in L^2(J)$. The result follows from the previous step by density of $H^2\big|_J$ in $L^2(J)$ along the line of reasoning similar to [7].

More precisely, by density (Proposition 1.2.3), for a given $h \in L^2(J)$, we have existence of a sequence $\{h_n\}_{n=1}^{\infty} \subset H^2\big|_J$ such that $h_n \underset{n\to\infty}{\to} h$ in $L^2(J)$. This generates a sequence of solutions

$$g_n = \arg\min_{g\in\mathcal{B}_{M,h_n}} \|\psi+bg-f\|_{L^2(I)}, \quad n\in\mathbb{N}_+, \qquad (1.28)$$

satisfying

$$(1+\mu_n\phi)\, g_n = P_+\left(\bar{b}\,(f-\psi)\vee(1+\mu_n)\,\bar{b}\,(h_n-\psi)\right) \qquad (1.29)$$

for $\mu_n > -1$ chosen such that $\|\psi+bg_n-h_n\|_{L^2(J)} = M$.

Since $\{g_n\}_{n=1}^{\infty}$ is bounded in H^2 (by definition of the solution space $\mathcal{B}^{\psi,b}_{M,h_n}$), and due to the Hilbertian setting, up to extraction of a subsequence, it converges weakly in $L^2(\mathbb{T})$ norm to some element in H^2

$$g_n \underset{n\to\infty}{\rightharpoonup} \gamma \in H^2. \qquad (1.30)$$

We will first show that $\mu_n \to \mu$ as $n\to\infty$. Then, since all $(1+\mu\phi)$ and $(1+\mu_n\phi)$ are self-adjoint, we have, for any $\xi \in H^2$,

$$\langle(1+\mu_n\phi)\, g_n,\xi\rangle_{L^2(\mathbb{T})} = \langle g_n,(1+\mu_n\phi)\,\xi\rangle_{L^2(\mathbb{T})} \underset{n\to\infty}{\to} \langle\gamma,(1+\mu\phi)\,\xi\rangle_{L^2(\mathbb{T})} = \langle(1+\mu\phi)\,\gamma,\xi\rangle_{L^2(\mathbb{T})},$$

and thus $(1+\mu_n\phi)\, g_n \underset{n\to\infty}{\rightharpoonup} (1+\mu\phi)\,\gamma$. Combining this with the convergence

$$P_+\left(\bar{b}\,(f-\psi)\vee(1+\mu_n)\,\bar{b}\,(h_n-\psi)\right) \underset{n\to\infty}{\to} P_+\left(\bar{b}\,(f-\psi)\vee(1+\mu)\,\bar{b}\,(h-\psi)\right)$$

in $L^2(\mathbb{T})$, equation (1.29) suggests that the weak limit γ in (1.30) is a solution to (1.17). It will remain to check that $\gamma \in \mathcal{B}^{\psi,b}_{M,h}$ and is indeed a minimizer of the cost functional (1.17).

Claim 1.3.1. *For μ_n in (1.29), we have*

$$\lim_{n\to\infty}\mu_n =: \mu \in (-1, \infty)\,. \tag{1.31}$$

Proof. We prove this statement by contradiction. Because of the relation (1.22), for any $\xi \in H^2$, we have

$$\left\langle \bar{b}\,(f-\psi) - g_n, \xi \right\rangle_{L^2(I)} = (1+\mu_n)\left\langle g_n - \bar{b}\,(h_n-\psi)\,, \xi \right\rangle_{L^2(J)}. \tag{1.32}$$

We note that the weak convergence (1.30) in H^2 implies the weak convergence $g_n \rightharpoonup \gamma$ in $L^2(J)$ as $n\to\infty$ since for a given $\eta \in L^2(J)$, we can take $\xi = P_+(0 \vee \eta) \in H^2$ in the definition $\lim\limits_{n\to\infty}\langle g_n, \xi\rangle_{L^2(\mathbb{T})} = \langle \gamma, \xi\rangle_{L^2(\mathbb{T})}$.

Assume first that $\mu_n \underset{n\to\infty}{\to} \infty$. Then, since the left-hand side of (1.32) remains bounded as $n\to\infty$, we necessarily must have

$$\lim_{n\to\infty}\left\langle g_n - \bar{b}\,(h_n-\psi)\,, \xi \right\rangle_{L^2(J)} = 0.$$

Since $h_n \to h$ in $L^2(J)$ strongly, this implies that $\gamma = \bar{b}\,(h-\psi) \in H^2\big|_J$ contrary to the initial assumption of the section that $h \notin H^2\big|_J$.

Next, we consider another possibility, namely that the limit $\lim\limits_{n\to\infty}\mu_n$ does not exist. Then, there are at least two infinite sequences $\{n_{k_1}\}$, $\{n_{k_2}\}$ such that

$$\lim_{k_1\to\infty}\mu_{n_{k_1}} =: \mu^{(1)} \neq \mu^{(2)} := \lim_{k_2\to\infty}\mu_{n_{k_2}}.$$

Since the left-hand side of (1.32) is independent of μ_n and both limits $\mu^{(1)}$, $\mu^{(2)}$ exist and finite, we have

$$\lim_{k_1\to\infty}\left(1+\mu_{n_{k_1}}\right)\left\langle g_{n_{k_1}} - \bar{b}\left(h_{n_{k_1}}-\psi\right), \xi \right\rangle_{L^2(J)} = \lim_{k_2\to\infty}\left(1+\mu_{n_{k_2}}\right)\left\langle g_{n_{k_2}} - \bar{b}\left(h_{n_{k_2}}-\psi\right), \xi \right\rangle_{L^2(J)}$$

$$\Rightarrow \quad \left(\mu^{(1)}-\mu^{(2)}\right)\left\langle \gamma - \bar{b}\,(h-\psi)\,, \xi \right\rangle_{L^2(J)} = 0.$$

As before, because of $h \notin H^2\big|_J$, we derive a contradiction $\mu^{(1)} = \mu^{(2)}$.

Now that the limit in (1.31) exists, we have $\mu \geq -1$. To show $\mu > -1$, assume, by contradiction, that $\mu = -1$. Since $g_n \in \mathcal{B}_{M,h_n}$, for any $\xi \in H^2$, the Cauchy-Schwarz inequality gives

$$\operatorname{Re}\langle \psi + b g_n - h_n, \xi\rangle_{L^2(J)} \geq -M\,\|\xi\|_{L^2(J)}\,,$$

and hence it follows from (1.32) (taking real part and passing to the limit as $n\to\infty$) that

$$\underbrace{-(1+\mu)\,M\,\|\xi\|_{L^2(J)}}_{=0} \leq \operatorname{Re}\langle f - \psi - b\gamma, \xi\rangle_{L^2(I)}\,,$$

which results in a contradiction since the right-hand side may be made negative due to the assumption that $f \notin \mathcal{A}^{\psi,b}\big|_I$ and to the arbitrary choice of ξ, whereas the left-hand side vanishes by the assumption $\mu = -1$. This finishes the proof of (1.31). $\square$

Claim 1.3.2. $\gamma \in \mathcal{B}^{\psi,b}_{M,h}$.

Proof. For $g_n \in \mathcal{B}^{\psi,b}_{M,h_n}$, we have $\|\psi + bg_n - h_n\|_{L^2(J)} \le M$. But $h_n \to h$ in $L^2(J)$, $g_n \rightharpoonup \gamma$ in $L^2(J)$ (as discussed in the proof of Claim 1) and so also $\psi + bg_n - h_n \rightharpoonup \psi + b\gamma - h$ in $L^2(J)$ as $n \to \infty$. The claim now is a direct consequence of the general property of weak limits:

$$\|\tilde{g}\| \le \liminf_{n\to\infty} \|\tilde{g}_n\| \quad \text{whenever} \quad \tilde{g}_n \rightharpoonup \tilde{g} \quad \text{as} \quad n \to \infty, \tag{1.33}$$

which follows from taking $\xi = \tilde{g}$ in $\lim_{n\to\infty} \langle \tilde{g}_n, \xi\rangle = \langle \tilde{g}, \xi\rangle$ and the Cauchy-Schwarz inequality.

□

Claim 1.3.3. *γ is a minimizer of (1.17).*

Proof. Since $\gamma \in \mathcal{B}^{\psi,b}_{M,h}$ and g_0 is a minimizer of (1.17), we have

$$\|\psi + bg_0 - f\|_{L^2(I)} \le \|\psi + b\gamma - f\|_{L^2(I)}.$$

To deduce the equality, by contradiction, we assume the strict inequality, or equivalently

$$\|\psi + bg_0 - f\|_{L^2(I)} \le \|\psi + b\gamma - f\|_{L^2(I)} - \xi \tag{1.34}$$

for some $\xi > 0$. We want to show that this inequality would lead to a contradiction between optimality of solutions $g_0 \in \mathcal{B}^{\psi,b}_{M,h}$ and $g_n \in \mathcal{B}^{\psi,b}_{M,h_n}$ for sufficiently large n.

First of all, there exists $g_0^\star \in \mathcal{B}^{\psi,b}_{M,h}$ and $\tau > 0$ such that

$$\|\psi + bg_0 - f\|_{L^2(I)} = \|\psi + bg_0^\star - f\|_{L^2(I)} - \tau \tag{1.35}$$

and $\|\psi + bg_0^\star - h\|_{L^2(J)} < M$. Indeed, take $g_0^\star = g_0 + \epsilon\delta_g$ with $\delta_g \in H^2$, $\|\delta_g\|_{H^2} = 1$ such that

$$\mathrm{Re}\,\langle \psi + bg_0 - h, b\delta_g\rangle_{L^2(J)} < 0. \tag{1.36}$$

Then, since $\|\psi + bg_0 - h\|_{L^2(J)} = M$ (according to Lemma 1.3.1), we have

$$\|\psi + bg_0^\star - h\|^2_{L^2(J)} = \|\psi + bg_0 - h\|^2_{L^2(J)} + 2\epsilon\mathrm{Re}\,\langle \psi + bg_0 - h, b\delta_g\rangle_{L^2(J)} + \epsilon^2 \|\delta_g\|^2_{L^2(J)} = M^2 - \eta_0$$

with $\eta_0 := -2\epsilon\mathrm{Re}\,\langle \psi + bg_0 - h, b\delta_g\rangle_{L^2(J)} - \epsilon^2 \|\delta_g\|^2_{L^2(J)} > 0$ for sufficiently small $\epsilon > 0$, that is

$$\|\psi + bg_0^\star - h\|_{L^2(J)} = M - \eta, \quad \eta := \frac{\eta_0}{\|\psi + bg_0^\star - h\|_{L^2(J)} + M} > 0. \tag{1.37}$$

Now we consider

$$\|\psi + bg_0^\star - f\|^2_{L^2(I)} = \|\psi + bg_0 - f\|^2_{L^2(I)} + 2\epsilon \mathrm{Re}\left\langle \psi + bg_0 - f, b\delta_g \right\rangle_{L^2(I)} + \epsilon^2 \|\delta_g\|^2_{L^2(I)}$$

and note that the optimality condition (1.22) implies

$$\left\langle \bar{b}\left(f-\psi\right) - g_0, \delta_g \right\rangle_{L^2(I)} = (1+\mu) \left\langle g_0 - \bar{b}\left(h-\psi\right), \delta_g \right\rangle_{L^2(J)}$$

$$\Rightarrow \quad \mathrm{Re}\left\langle \psi + bg_0 - f, b\delta_g \right\rangle_{L^2(I)} = -\left(1+\mu\right) \mathrm{Re}\left\langle \psi + bg_0 - h, b\delta_g \right\rangle_{L^2(J)} > 0,$$

where $\mu > -1$ is the Lagrange parameter for the solution g_0. Therefore,

$$\|\psi + bg_0^\star - f\|^2_{L^2(I)} = \|\psi + bg_0 - f\|^2_{L^2(I)} + \tau_0$$

with $\tau_0 := -2\left(1+\mu\right) \mathrm{Re}\left\langle \psi + bg_0 - h, b\delta_g \right\rangle_{L^2(J)} + \epsilon^2 \|\delta_g\|^2_{L^2(I)} > 0$ for small enough ϵ, and so (1.35) follows with

$$\tau := \frac{\tau_0}{\|\psi + bg_0^\star - f\|_{L^2(I)} + \|\psi + bg_0 - f\|_{L^2(I)}} > 0. \tag{1.38}$$

Now it is easy to see that for large enough n, we also have $g_0^\star \in \mathcal{B}^{\psi,b}_{M,h_n}$. Since $h_n \to h$ in $L^2\left(J\right)$ as $n \to \infty$, there exists $N_1 \in \mathbb{N}_+$ such that $\|h - h_n\|_{L^2(J)} < \eta$ whenever $n > N_1$, so from (1.37), we deduce the bound

$$\|\psi + bg_0^\star - h_n\|_{L^2(J)} \leq \|\psi + bg_0^\star - h\|_{L^2(J)} + \|h - h_n\|_{L^2(J)} \leq M. \tag{1.39}$$

On the other hand, by the property of weak limits (1.33), we have

$$\liminf_{n\to\infty} \|\psi + bg_n - f\|_{L^2(I)} \geq \|\psi + b\gamma - f\|_{L^2(I)},$$

that is, for any given $\rho > 0$,

$$\|\psi + bg_n - f\|_{L^2(I)} > \|\psi + b\gamma - f\|_{L^2(I)} - \rho \tag{1.40}$$

holds when n is taken large enough. In particular, there is $N_2 \in \mathbb{N}_+$ such that (1.40) holds for $n \geq N_2$ with $\rho = \tau$. Then, for any $n \geq \max\left\{N_1, N_2\right\}$, (1.40) can be combined with (1.34) and (1.35) to give

$$\|\psi + bg_n - f\|_{L^2(I)} > \|\psi + bg_0^\star - f\|_{L^2(I)} + \xi - 2\tau.$$

According to (1.38), τ can be made arbitrarily small by the choice of δ_g and ϵ whereas ξ is a fixed number. Therefore, we have $\|\psi + bg_0^\star - f\|_{L^2(I)} < \|\psi + bg_n - f\|_{L^2(I)}$ and $g_0^\star \in \mathcal{B}^{\psi,b}_{M,h_n}$ (according to (1.39)). In other words, $g_0^\star$ gives a better solution than g_n, and hence, by uniqueness (Theorem 1.3.1), we get a contradiction to the minimality of g_n in (1.28). □

Remark 1.3.1. *As it is mentioned in the formulation of Theorem 1.3.2, for g_0 to be a solution to (1.17), the*

Lagrange parameter μ has yet to be chosen such that g_0 given by (1.23) satisfies the constraint $\|\psi + bg_0 - h\|_{L^2(J)} = M$, which makes the well-posedness (regularization) effective, see Proposition 1.2.4 and discussion in the beginning of Section 1.5.

We note that the formal substitution $\mu = -1$ in (1.27) leaves out the constraint on J and leads to the situation $g|_I = \bar{b}(f-\psi)$ that was ruled out initially by the requirement (1.19).

When $f \in \mathcal{A}^{\psi,b}\big|_I$, we face an extrapolation problem of holomorphic extension from I inside the disk preserving interior pointwise data. In such a case, $\bar{b}(f-\psi) \in H^2\big|_I$ and Proposition 1.2.1 (or alternative scheme from [37] mentioned in Remark 1.2.3) applies to construct the extension g_0 such that $g_0|_I = \bar{b}(f-\psi)$ which can be regarded as the solution if we give up the control on J which means that for a given h the parameter M should be relaxed (yet remaining finite) to avoid an overdetermined problem. Otherwise, keeping the original bound M, despite $f \in \mathcal{A}^{\psi,b}\big|_I$, we must accept non-zero minimum of the cost functional of the problem in which case the solution g_0 is still given by (1.23) which proof is valid since now $g_0|_I \neq \bar{b}(f-\psi)$. The latter situation, from geometrical point of view, is nothing but finding a projection of $f \in \mathcal{A}^{\psi,b}\big|_I$ onto the convex subset $\mathcal{C}^{\psi,b}_{M,h} \subseteq \mathcal{A}^{\psi,b}\big|_I$.

However, returning back to the realistic case, when $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$, the solution to (1.17) can still be written in an integral form in spirit of the Carleman's formula (1.12) as given by the following result (see also [7] where it was stated for the case $\psi \equiv 0$, $b \equiv 1$).

Proposition 1.3.1. *For $\mu \in (-1, 0)$, the solution (1.23) can be represented as*

$$g_0(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \left(\frac{\Phi(\xi)}{\Phi(z)}\right)^{\alpha} \left(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)\right)(\xi)\,\frac{d\xi}{\xi - z}, \quad z \in \mathbb{D}, \tag{1.41}$$

where

$$\Phi(z) = \exp\left\{\frac{\log\rho}{2\pi i}\int_I \frac{\xi+z}{\xi-z}\frac{d\xi}{\xi}\right\}, \quad \alpha = -\frac{\log(1+\mu)}{2\log\rho}, \quad \rho > 1. \tag{1.42}$$

Proof. First of all, we note that (1.42) is a quenching function satisfying $|\Phi| = \rho \vee 1$ on $\mathbb{T}$ and $|\Phi| > 1$ on $\mathbb{D}$ which can be constructed following the recipe of Remark 1.2.2. The condition $|\Phi| > 1$ on $\mathbb{D}$ and the minimum modulus principle for analytic functions imply the requirement $\rho > 1$.

To show the equivalence, one can start from (1.41) and arrive at (1.23) for a suitable choice of the parameters. Indeed, since $\Phi \in H^\infty$, (1.41) implies

$$\Phi^\alpha g_0 = P_+\left[\Phi^\alpha\left(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)\right)\right]$$

$$\Rightarrow \quad P_+\left(|\Phi|^{2\alpha} g_0\right) = P_+\left(\bar{\Phi}^\alpha P_+\left[\Phi^\alpha\left(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)\right)\right]\right).$$

We can represent

$$P_+\left[\Phi^\alpha\left(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)\right)\right] = \Phi^\alpha\left(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)\right) - P_-\left[\Phi^\alpha\left(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)\right)\right]$$

with P_- being anti-analytic projection defined in Section 1.2. Since

$$\left\langle \bar{\Phi}^\alpha P_- \left[\Phi^\alpha \left(\bar{b}\left(f-\psi\right) \vee \bar{b}\left(h-\psi\right)\right)\right], u\right\rangle_{L^2(\mathbb{T})} = \left\langle P_- \left[\Phi^\alpha \left(\bar{b}\left(f-\psi\right) \vee \bar{b}\left(h-\psi\right)\right)\right], \Phi^\alpha u\right\rangle_{L^2(\mathbb{T})} = 0$$

for any $u \in H^2$, it follows that $P_+\left(\bar{\Phi}^\alpha P_- \left[\Phi^\alpha \left(\bar{b}\left(f-\psi\right) \vee \bar{b}\left(h-\psi\right)\right)\right]\right) = 0$ and so we deduce

$$P_+ \left(\left|\Phi\right|^{2\alpha} g_0\right) = P_+ \left[\left|\Phi\right|^{2\alpha} \left(\bar{b}\left(f-\psi\right) \vee \bar{b}\left(h-\psi\right)\right)\right].$$

Given $\rho > 1$, choose $\alpha > 0$ such that $\rho^{2\alpha} = \dfrac{1}{1+\mu}$ (this restricts the range $\mu > -1$ to $\mu \in (-1, 0)$). Then, $\left.\left|\Phi\right|^{2\alpha}\right|_I = \dfrac{1}{1+\mu}$, $\left.\left|\Phi\right|^{2\alpha}\right|_J = 1$, and hence (1.23) is furnished since

$$P_+ \left(\frac{1}{1+\mu} g_0 \vee g_0\right) = P_+ \left(\frac{\bar{b}}{1+\mu}\left(f-\psi\right) \vee \bar{b}\left(h-\psi\right)\right)$$

$$\Rightarrow \quad P_+\left(g_0 \vee g_0\right) + \mu P_+\left(0 \vee g_0\right) = P_+ \left(\bar{b}\left(f-\psi\right) \vee \left(1+\mu\right)\bar{b}\left(h-\psi\right)\right).$$

□

Remark 1.3.2. *We would also like to point put an alternative path leading to the same representation of the solution which may be looked as a new way to derive a Carleman formula (1.12) using a combination of solutions of bounded extremal problem and Riemann-Hilbert problem for a disk that we are going to formulate now. Rewrite (1.23) as*

$$\left(1 + \mu\chi_J(z)\right) g_0(z) = L(z) + R(z), \quad z \in \mathbb{T},$$

where

$$L(z) := \bar{b}\left(f-\psi\right) \vee \left(1+\mu\right)\bar{b}\left(h-\psi\right),$$

and $R \in H^2_-$ is an unknown function. Equivalently,

$$g_0(z) = G(z) R(z) + L_0(z), \quad z \in \mathbb{T}, \tag{1.43}$$

where

$$G(z) := \frac{1}{1+\mu\chi_J(z)}, \quad L_0(z) := L(z) G(z),$$

which can be viewed as a conjugation (Riemann-Hilbert) problem for holomorphic functions inside and outside the disk $\mathbb{D}$, namely, g_0 and R (see [22]). To solve it, we need to factorize its coefficient as

$$G(z) = G_+(z) / G_-(z), \quad z \in \mathbb{T},$$

where G_+ and G_- are traces of functions analytic inside and outside $\mathbb{D}$, respectively. In order to construct this factorization, we use the fact that G is non-vanishing on $\mathbb{T}$ and observe that functions $\log G_+$ and $\log G_-$ have the *same domains of analyticity as G_+ and G_-. Then, upon taking logarithms, we achieve the decomposition*

$$\log G = \log G_+ - \log G_-$$

employing classical Plemelj-Sokhotskii formulas which yield

$$\log G_\pm(z) = \lim_{|z|\to 1-0^\pm} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log G(\tau)}{\tau - z} d\tau = \mp\frac{\log(1+\mu)}{2}\chi_J(z) - \frac{\log(1+\mu)}{2\pi i} \fint_J \frac{d\tau}{\tau - z},$$

and hence

$$G_\pm(z) = \exp\left[\mp\frac{\log(1+\mu)}{2}\chi_J(z) - \frac{\log(1+\mu)}{2\pi i}\fint_J \frac{d\tau}{\tau - z}\right].$$

Now, from (1.43), *we have*

$$g_0/G_+ - R/G_- = L_0/G_+,$$

where in the left-hand side we have the difference between H^2_+ and H^2_- functions. Comparing them with boundary values of the Cauchy integral

$$Y(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{L_0(\tau)}{G_+(\tau)}\frac{d\tau}{\tau - z}, \tag{1.44}$$

on $\mathbb{T}$ obtained again by Plemelj-Sokhotskii formulas

$$Y_\pm(z) := \lim_{|z|\to 1-0^\pm}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{L_0(\tau)}{G_+(\tau)}\frac{d\tau}{\tau - z} = \pm\frac{L_0(z)}{2G_+(z)} + \frac{1}{2\pi i}\fint_{\mathbb{T}}\frac{L_0(\tau)}{G_+(\tau)}\frac{d\tau}{\tau - z},$$

we arrive at

$$\frac{g_0}{G_+} - Y_+ = \frac{R}{G_-} - Y_-.$$

The last equality means that both left- and right-hand sides are restrictions of a single entire function E. Since $R \in H^2_-$, R vanishes at infinity as well as the Cauchy integral (1.44)*. Altogether this implies vanishing of the entire function E, and Liouville theorem [1] then asserts that $E \equiv 0$ in $\mathbb{C}$. Therefore, we deduce that*

$$g_0(z) = G_+(z)\,Y_+(z), \quad z \in \mathbb{T},$$

and, for $z \in \mathbb{D}$,

$$g_0(z) = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{L_0(\tau)\,G_+(z)}{G_+(\tau)}\frac{d\tau}{\tau - z} = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{L(\tau)\,G_+(z)}{G_-(\tau)}\frac{d\tau}{\tau - z}. \tag{1.45}$$

Employing the identities

$$\int_J \frac{d\xi}{\xi - z} = 2\pi i - \int_I \frac{d\xi}{\xi - z}, \quad z \in \mathbb{D}, \qquad \fint_J \frac{d\xi}{\xi - \tau} = \pi i - \fint_I \frac{d\xi}{\xi - \tau}, \quad \tau \in \mathbb{T},$$

(1.45) *can be rewritten as*

$$g_0(z) = \frac{1}{2\pi i}\int_{\mathbb{T}}\left(\frac{\bar{b}(f-\psi)}{\sqrt{1+\mu}} \vee \bar{b}(h-\psi)\right)(\tau)\exp\left[\frac{\log(1+\mu)}{2\pi i}\left(\int_I\frac{d\xi}{\xi - z} - \fint_I\frac{d\xi}{\xi - \tau}\right)\right]\frac{d\tau}{\tau - z}$$

which coincides with (1.41) after the use of Plemelj-Sokhotskii formulas to compute boundary values from inside $\mathbb{D}$ *in (1.42) when* $z \in \mathbb{T}$.

1.4 Choice of interpolation function and solution reduction

Before we proceed with computational aspects, it is worth discussing the choice of interpolant ψ which up to this point was any H^2 function satisfying the interpolation conditions (1.13).

We will first consider a particular choice of the interpolant following [41] and then discuss the general case.

Proposition 1.4.1. *The* H^2 *function defined for* $z \in \mathbb{D}$ *by*

$$\psi(z) = \sum_{k=1}^{N} \psi_k \mathcal{K}(z_k, z) \quad \text{with} \quad \mathcal{K}(z_k, z) := \frac{1}{1 - \bar{z}_k z} \tag{1.46}$$

interpolates the data (1.13) for an appropriate choice of the constants $\{\psi_k\}_{k=1}^N \in \mathbb{C}$ *which exists regardless of a priori prescribed values* $\{\omega_k\}_{k=1}^N$ *and choice of the points* $\{z_k\}_{k=1}^N$ *(providing they are all different). Moreover, it is the unique interpolant of minimal* H^2 *norm.*

Proof. We note that the function $\mathcal{K}(\cdot,\cdot)$ is the reproducing kernel for H^2 meaning that, for any $u \in H^2$, $z_0 \in \mathbb{D}$, point evaluation is given by the inner product $u(z_0) = \langle u, \mathcal{K}(z_0, \cdot)\rangle_{L^2(\mathbb{T})}$, which is a direct consequence of the Cauchy integral formula because $d\theta = \dfrac{1}{iz}dz$ in (1.8). The coefficients $\{\psi_k\}_{k=1}^N \in \mathbb{C}$ in (1.46) are to be found from the requirement (1.13). We therefore have

$$\psi_k = \sum_{j=1}^{N} S_{kj}\omega_j, \quad \text{where} \quad S := [S_{kj}] = [\mathcal{K}(z_k, z_j)]^{-1}\,, \ k, j = 1, \dots, N. \tag{1.47}$$

In order to see that the existence of the inverse matrix S is unconditional, we note that $\mathcal{K}(z_k, z_j) = \langle \mathcal{K}(z_k, \cdot), \mathcal{K}(z_j, \cdot)\rangle_{L^2(\mathbb{T})}$, and hence it is the inverse of a Gram matrix which exists since $z_k \neq z_j$ whenever $k \neq j$ providing that all functions $\{\mathcal{K}(z_k, z)\}_{k=1}^N$ are linearly independent. To check the latter, we verify the implication

$$\sum_{k=1}^{N} c_k \mathcal{K}(z_k, z) = 0 \qquad \Rightarrow \qquad c_k = 0, \quad k = 1, \dots, N.$$

Employing the identity $\dfrac{1}{1 - \bar{z}_k z} = \sum_{n=0}^{\infty} \bar{z}_k^n z^n$ that holds due to $|z_k z| < 1$, we see that

$$\sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k \bar{z}_k^n \right) z^n = 0, \quad \forall z \in \mathbb{D} \qquad \Rightarrow \qquad \sum_{k=1}^{N} c_k \bar{z}_k^n = 0, \quad n \in \mathbb{N}_0.$$

But, by induction on k, this necessarily implies that $c_k = 0$, $k = 1, \dots, N$ and thus proves the linear independence.

To show that $\psi \in H^2$ is the unique interpolant of minimal norm, we let $\psi_0 \in H^2$ be another interpolant satisfying (1.13). Then, $\phi_0 := \psi - \psi_0 \in H^2$ is such that $\phi_0|_{z=z_k} = 0$, $k = 1, \dots, N$, or equivalently,

$$\langle \phi_0, \mathcal{K}(z_k, \cdot)\rangle_{L^2(\mathbb{T})} = 0, \quad k = 1, \dots, N$$

meaning orthogonality of $\phi_0(z)$ to a linear span of $\{\mathcal{K}(z_k, z)\}_{k=1}^N$. But ψ exactly belongs to this span, and hence

$$\|\psi_0\|_{H^2}^2 = \|\psi\|_{H^2}^2 + \|\phi_0\|_{H^2}^2 > \|\psi\|_{H^2}^2, \tag{1.48}$$

which shows that ψ is the unique interpolating H^2 function of minimal norm. □

Remark 1.4.1. *With this choice of ψ, the solution (1.23) takes the form*

$$g_0 = (1+\mu\phi)^{-1}\left[P_+\left(\bar{b}\,(f\vee h)\right) + \mu\, P_+\left(0\vee \bar{b}\,(h-\psi)\right)\right]. \tag{1.49}$$

Indeed, since $\langle \mathcal{K}(z_k, z), bu\rangle_{L^2(\mathbb{T})} = 0$, $k = 1,\dots,N$ for any $u \in H^2$, we have $P_+\left(\bar{b}\psi\right) = 0$ whenever ψ is given by (1.46).

Now it may look tempting to consider other choices of the interpolant to improve the L^2-bounds in (1.15) or (1.17) rather than being itself of minimal $L^2(\mathbb{T})$ norm. However, the choice of the interpolant does not affect the combination $\tilde{g}_0 = \psi + bg_0$, a result that is not surprising at all from physical point of view since ψ is an auxiliary tool which should not affect solution whose dependence must eventually boil down to given data (measurement related quantities) only: $\{z_k\}_{k=1}^N$, $\{\omega_k\}_{k=1}^N$, f and h. More precisely, we have

Lemma 1.4.1. *Given arbitrary ψ_1, $\psi_2 \in H^2$ satisfying (1.13), we have $\psi_1 + bg_0(\psi_1) = \psi_2 + bg_0(\psi_2)$.*

Proof. First of all, we note that the dependence $g_0(\psi)$ is not only due to explicit appearance of ψ in (1.23), but also because the Lagrange parameter μ, in general, has to be readjusted according to ψ, that is $\mu = \mu(\psi)$ so that

$$\|\psi_k + bg_0(\psi_k) - h\|_{L^2(J)}^2 = M^2, \quad k = 1, 2, \tag{1.50}$$

where we mean $g_0(\psi) = g_0(\psi, \mu(\psi))$. Let us denote $\delta_\psi := \psi_2 - \psi_1$, $\delta_\mu := \mu(\psi_2) - \mu(\psi_1)$, $\delta_g := g_0(\psi_2) - g_0(\psi_1)$. Taking difference of both equations (1.50), we have

$$\langle \delta_\psi + b\delta_g, \psi_1 + bg_0(\psi_1) - h\rangle_{L^2(J)} + \langle \psi_2 + bg_0(\psi_2) - h, \delta_\psi + b\delta_g\rangle_{L^2(J)} = 0$$

$$\Rightarrow \quad 2\mathrm{Re}\left\langle \bar{b}\delta_\psi + \delta_g, \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h\right\rangle_{L^2(J)} = \|\delta_\psi + b\delta_g\|_{L^2(J)}^2. \tag{1.51}$$

On the other hand, the optimality condition (1.20) implies that, for any $\xi \in H^2$,

$$\left\langle \bar{b}\psi_k + g_0(\psi_k) - \bar{b}f, \xi\right\rangle_{L^2(I)} = -\left(1+\mu(\psi_k)\right)\left\langle \bar{b}\psi_k + g_0(\psi_k) - \bar{b}h, \xi\right\rangle_{L^2(J)}, \quad k = 1, 2,$$

and therefore

$$\left\langle \bar{b}\delta_\psi + \delta_g, \xi\right\rangle_{L^2(I)} = -\left(1+\mu(\psi_1)\right)\left\langle \bar{b}\delta_\psi + \delta_g, \xi\right\rangle_{L^2(J)} - \delta_\mu\left\langle \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h, \xi\right\rangle_{L^2(J)}. \tag{1.52}$$

Since $\delta_\psi \in H^2$, due to (1.13), it is zero at each z_j, $j = 1, \dots, N$, and hence factorizes as $\delta_\psi = b\eta$ for some $\eta \in H^2$. This allows us to take $\xi = \bar{b}\delta_\psi + \delta_g \in H^2$ in (1.52) to yield

$$\|\eta + \delta_g\|^2_{L^2(I)} = -\left(1 + \mu\left(\psi_1\right)\right)\|\eta + \delta_g\|^2_{L^2(J)} - \delta_\mu \left\langle \bar{b}\psi_2 + g_0\left(\psi_2\right) - \bar{b}h, \eta + \delta_g \right\rangle_{L^2(J)}.$$

Note that the inner product term here is real-valued since the others are, and so employing (1.51), we arrive at

$$\|\eta + \delta_g\|^2_{L^2(I)} + \left(1 + \mu\left(\psi_1\right)\right)\|\eta + \delta_g\|^2_{L^2(J)} = -\frac{1}{2}\delta_\mu \|\eta + \delta_g\|^2_{L^2(J)}$$

which, due to $\mu > -1$, entails that $\delta_\mu \leq 0$. But, clearly, interchanging ψ_1 and ψ_2, we would get $\delta_\mu \geq 0$, and so $\delta_\mu = 0$ leading to $\|\delta_\psi + b\delta_g\|^2_{L^2(\mathbb{T})} = \|\eta + \delta_g\|^2_{L^2(I)} + \|\eta + \delta_g\|^2_{L^2(J)} = 0$ which finishes the proof. □

Combining this lemma with Remark 1.4.1, we can formulate

Corollary 1.4.1. *Independently of choice of $\psi \in H^2$ fulfilling (1.13), the final solution $\tilde{g}_0 = \psi + bg_0$ is given by*

$$\tilde{g}_0 = \psi + b\left(1 + \mu\phi\right)^{-1}\left[P_+\left(\bar{b}\left(f \vee h\right)\right) + \mu P_+\left(0 \vee \bar{b}\left(h - \psi\right)\right)\right]. \tag{1.53}$$

These results will be employed for analytical purposes in Section 1.7.

Even though it is not going to be used here, we also note that it is possible to construct an interpolant whose norm does not exceed *a priori* given bound providing a certain quadratic form involving interpolation data and value of the bound is positive semidefinite [19].

1.5 Computational issues and error estimate

We would like to stress again that the obtained formulas (1.23), (1.41) and (1.49) furnish solution only in an implicit form with the Lagrange parameter μ still to be chosen such that the solution satisfies the equality constraint in (1.15). As it was mentioned in Remark 1.3.1, the constraint in $\mathcal{B}^{\psi,b}_{M,h}$ does not enter the solution characterisation (1.27) when $\mu = -1$, so as $\mu \searrow -1$ we expect perfect approximation of the given $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$ at the expense of uncontrolled growth of the quantity

$$M_0(\mu) := \|\psi + bg_0(\mu) - h\|_{L^2(J)} \tag{1.54}$$

according to Propositions 1.2.3 and 1.2.4. This is not surprising since the inclusion $\mathcal{B}^{\psi,b}_{M_1,h} \subset \mathcal{B}^{\psi,b}_{M_2,h}$ whenever $M_1 < M_2$ implies that the minimum of the cost functional of (1.17) sought over $\mathcal{B}^{\psi,b}_{M_1,h}$ is bigger than that for $\mathcal{B}^{\psi,b}_{M_2,h}$. For devising a solution feasible for applications, a suitable trade-off between value of μ (governing quality of approximation on I) and choice of the admissible bound M has to be found. To gain insight into this situation, we define the error of approximation as

$$e(\mu) := \|\psi + bg_0(\mu) - f\|^2_{L^2(I)}, \tag{1.55}$$

and proceed with establishing connection between e and M_0.

1.5.1 Monotonicity and boundedness

Here we mainly follow the steps of [5, 7] where similar studies has been done without interpolation conditions.

Proposition 1.5.1. *The following monotonicity results hold*

$$\frac{de}{d\mu} > 0, \qquad \frac{dM_0^2}{d\mu} < 0. \tag{1.56}$$

Moreover, we have

$$\frac{de}{d\mu} = -(\mu+1)\frac{dM_0^2}{d\mu}. \tag{1.57}$$

Proof. From (1.23), using commutation of ϕ and $(1+\mu\phi)^{-1}$, we compute

$$\frac{dg_0}{d\mu} = -(1+\mu\phi)^{-2}\phi P_+\left(\bar{b}(f-\psi)\vee(1+\mu)\bar{b}(h-\psi)\right) + (1+\mu\phi)^{-1}P_+\left(0\vee\bar{b}(h-\psi)\right)$$

$$\Rightarrow \quad \frac{dg_0}{d\mu} = -(1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0\vee\bar{b}(\psi-h)\right)\right], \tag{1.58}$$

and thus

$$\frac{dM_0^2}{d\mu} = 2\mathrm{Re}\left\langle b\frac{dg_0}{d\mu}, \psi + bg_0 - h\right\rangle_{L^2(J)}$$

$$= -2\mathrm{Re}\left\langle(1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0\vee\bar{b}(\psi-h)\right)\right], \phi g_0 + P_+\left(0\vee\bar{b}(\psi-h)\right)\right\rangle_{L^2(\mathbb{T})} < 0, \tag{1.59}$$

The inequality here is due to the spectral result (1.25) implying

$$\mathrm{Re}\left\langle(1+\mu\phi)^{-1}\xi,\xi\right\rangle_{L^2(\mathbb{T})} = \left\langle(1+\mu\phi)^{-1}\xi,\xi\right\rangle_{L^2(\mathbb{T})} \geq 0$$

for any $\xi \in H^2$ and $\mu > -1$ whereas the equality in (1.59) would be possible, according to Proposition 1.2.2, only when $g_0|_J = \bar{b}(h-\psi)$, that is $M_0 = 0$, the case that was eliminated by Corollary 1.3.1.

Now, for any $\beta \in \mathbb{R}$, making use of (1.58) again, we compute

$$\begin{aligned}
\frac{de}{d\mu} &= 2\mathrm{Re}\left\langle\frac{dg_0}{d\mu}, \bar{b}(\psi-f) + g_0\right\rangle_{L^2(I)} \\
&= -2\mathrm{Re}\left\langle(1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0\vee\bar{b}(\psi-h)\right)\right], \left(\bar{b}(\psi-f)+g_0\right)\vee 0\right\rangle_{L^2(\mathbb{T})} \\
&= -\beta\frac{dM_0^2}{d\mu} - 2\mathrm{Re}B,
\end{aligned}$$

with B given by

$$\begin{aligned}
&\left\langle (1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0\vee \bar{b}\left(\psi-h\right)\right)\right], \beta\phi g_0 + \beta P_+\left[0\vee \bar{b}\left(\psi-h\right)\right] + \left(\bar{b}\left(\psi-f\right)+g_0\right)\vee 0\right\rangle_{L^2(\mathbb{T})}\\
&= \left\langle (1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0\vee \bar{b}\left(\psi-h\right)\right)\right], \left(\bar{b}\left(\psi-f\right)+g_0\right)\vee \beta\left[\bar{b}\left(\psi-h\right)+g_0\right]\right\rangle_{L^2(\mathbb{T})}\\
&= \left\langle b\,(1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0\vee \bar{b}\left(\psi-h\right)\right)\right], \left(\psi+bg_0-f\right)\vee \beta\left(\psi+bg_0-h\right)\right\rangle_{L^2(\mathbb{T})},
\end{aligned}$$

where we suppressed the P_+ operator on the right part of the inner product in the second line due to the fact that the left part of it belongs to H^2.

The choice $\beta = \mu + 1 = \lambda$ entails $\mathrm{Re}B = 0$ due to (1.20), and we thus obtain (1.57). Since $\mu + 1 > 0$, (1.57) combines with (1.59) to furnish the remaining inequality in (1.56). □

In particular, equation (1.57) encodes how the decay of the approximation error on I is accompanied by $\tilde{g}_0 = \psi + bg_0$ departing further away from given h on J as $\mu \searrow -1$. Even though more concrete asymptotic estimates on the increase of $M_0(\mu)$ near $\mu = -1$ will be discussed later on, we start providing merely a rough square-integrability result which is contained in the following

Proposition 1.5.2. *The deviation M_0 of the solution $\tilde{g}_0$ from h on J has moderate growth as $\mu \searrow -1$ so that, for any $-1 < \mu_0 < \infty$,*

$$\int_{-1}^{\mu_0} M_0^2(\mu)\, d\mu < \infty. \tag{1.60}$$

Proof. Integration of (1.57) by parts from μ to μ_0 yields

$$e(\mu_0) - e(\mu) = (\mu+1)\, M_0^2(\mu) - (\mu_0+1)\, M_0^2(\mu_0) + \int_{\mu}^{\mu_0} M_0^2(\tau)\, d\tau. \tag{1.61}$$

As it was already mentioned in the beginning of the section, Proposition 1.2.3 implies that the cost functional goes to 0 when μ decays to -1:

$$e(\mu) \searrow 0 \quad \text{as} \quad \mu \searrow -1. \tag{1.62}$$

We are now going to estimate the behavior of the product $(\mu+1)\, M_0^2(\mu)$. First of all, since the constraint is saturated (Lemma 1.3.1), condition (1.22) implies that

$$\begin{aligned}
\langle f-\psi-bg_0, bg_0\rangle_{L^2(I)} &= (1+\mu)\,\langle h-\psi-bg_0, -bg_0\rangle_{L^2(J)}\\
&= (1+\mu)\, M_0^2 - (1+\mu)\,\langle h-\psi-bg_0, h-\psi\rangle_{L^2(J)},
\end{aligned} \tag{1.63}$$

and therefore

$$e^{1/2}(\mu)\, \|g_0\|_{L^2(I)} \geq \left|\langle f-\psi-bg_0, bg_0\rangle_{L^2(I)}\right| \geq (1+\mu)\, M_0\left(M_0 - \|h-\psi\|_{L^2(J)}\right).$$

Now, since $M_0 \nearrow \infty$ as $\mu \searrow -1$ (because of (1.62) and Proposition 1.2.3), the first term is dominant, and thus the right-hand side remains positive. Then, because of (1.62) and finiteness of $\|g_0\|_{L^2(I)}$ (by the triangle inequality,

$\|g_0\|_{L^2(I)} \leq e^{1/2}(\mu) + \|\psi - f\|_{L^2(I)}$), we conclude that

$$(\mu+1)\, M_0^2 \searrow 0 \quad \text{as} \quad \mu \searrow -1, \tag{1.64}$$

which allows us to deduce (1.60) from (1.61). □

Remark 1.5.1. *In the simplified case with no pointwise interpolation conditions (or those of zero-values) and no information on J, the conclusion of the Proposition can be strengthened to*

$$\|M_0\|_{L^2(-1,\infty)} := \left(\int_{-1}^{\infty} M_0^2(\mu)\, d\mu\right)^{1/2} = \|f\|_{L^2(I)}, \tag{1.65}$$

a result that was given in [5]. This mainly relies on the fact that, for $\psi \equiv 0$ and $h \equiv 0$,

$$g_0 \to 0 \quad \mathit{in} \quad L^2(\mathbb{T}) \quad \mathit{as} \quad \mu \nearrow \infty, \tag{1.66}$$

which holds by the following argument. Denoting $\tilde{f} := P_+\left(\bar{b}f \vee 0\right)$, the solution formulas (1.23) and (1.27) become $g_0 = (1+\mu\phi)^{-1}\tilde{f}$ and $\mu\phi g_0 = \tilde{f} - g_0$, respectively. From these, as $\mu \nearrow \infty$, using the spectral theorem (see Appendix), we obtain

$$\|\phi g_0\|_{H^2} = \frac{1}{\mu}\left\|\tilde{f} - g_0\right\|_{H^2} \leq \frac{1}{\mu}\|f\|_{L^2(I)}\left[1 + \left\|(1+\mu\phi)^{-1}\right\|\right] \leq \frac{2}{\mu}\|f\|_{L^2(I)} \searrow 0,$$

and hence, by Proposition 1.2.2, conclude that $\|g_0\|_{H^2} \searrow 0$. We also need to show that

$$(\mu+1)\, M_0^2 \searrow 0 \quad \mathit{as} \quad \mu \nearrow \infty, \tag{1.67}$$

but this follows from the positivity $(\mu+1)\, M_0^2 > 0$ and the observation that, for large enough μ, we have

$$\frac{d\left[(\mu+1)\, M_0^2\right]}{d\mu} = M_0^2 + (\mu+1)\frac{dM_0^2}{d\mu} < 0$$

(the inequality holds since, due to (1.59), the second term in the right-hand side is strictly negative whereas the first one goes to zero as μ increases). Finally, further elaboration of (1.63) into

$$e(\mu) + (1+\mu)\, M_0^2(\mu) = \langle \psi + bg_0 - f, \psi - f\rangle_{L^2(I)} + (1+\mu)\langle \psi + bg_0 - h, \psi - h\rangle_{L^2(J)}$$

yields, in the case $\psi \equiv 0$, $h \equiv 0$,

$$e(\mu) + (1+\mu)\, M_0^2(\mu) = \langle f - bg_0, f\rangle_{L^2(I)},$$

which, by (1.66)-(1.67), furnishes $\lim_{\mu\to\infty} e(\mu) = \|f\|_{L^2(I)}^2$, and hence (1.65) follows from (1.61) recalling again (1.62) and (1.64).

1.5.2 Sharper estimates

Precise asymptotic estimates near $\mu = -1$ were obtained in [8] using concrete spectral theory of Toeplitz operators [38, 39]. Namely, under some specific regularity assumptions on the boundary data f (related to integrability of the first derivative on I), we have

$$M_0^2(\mu) = \mathcal{O}\left((1+\mu)^{-1}\log^{-2}(1+\mu)\right), \quad e(\mu) = \mathcal{O}\left(\left|\log^{-1}(1+\mu)\right|\right) \qquad \text{as} \quad \mu \searrow -1. \tag{1.68}$$

Here we suggest a way of *a priori* estimation of approximation rate and error bounds without resorting to an iterative solution procedure. This is based on a Neumann-like expansion of the inverse Toeplitz operator which provides series representations for the quantities $e(\mu)$ and $M_0^2(\mu)$ for values of μ moderately greater than -1 and, therefore, complements previously obtained estimates of the asymptotic behavior of these quantities in the vicinity of $\mu = -1$. Moreover, using these series expansions, we further attempt to recover the estimates (1.68) without having concrete spectral theory involved, yet still appealing to some general spectral theory results.

It is convenient to introduce the quantity

$$\xi(\mu) := \phi g_0(\mu) + P_+\left(0 \vee \bar{b}(\psi - h)\right) \tag{1.69}$$

that enters equation (1.58). The main results will be obtained in terms of

$$\xi_0 := \xi(0) = \phi\left(P_+\left(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)\right)\right) - P_+\left(0 \vee \bar{b}(h-\psi)\right). \tag{1.70}$$

Proposition 1.5.3. *For $|\mu| < 1$, we have*

$$M_0^2(\mu) = M_0^2(0) - \sum_{k=0}^{\infty}(-1)^k \mu^{k+1}(k+2)F(k), \tag{1.71}$$

$$e(\mu) = e(0) + \sum_{k=0}^{\infty}(-1)^k \mu^{k+1}\left[(k+2)F(k) - kF(k-1)\right], \tag{1.72}$$

where $F(k) := \left\langle \phi^k \xi_0, \xi_0 \right\rangle_{L^2(\mathbb{T})}$, $k \in \mathbb{N}_+$.

Proof. Consider, for $k \in \mathbb{N}_+$, $\mu > -1$,

$$A_k(\mu) := \left\langle (1+\mu\phi)^{-k}\phi^{k-1}\xi(\mu), \xi(\mu)\right\rangle_{L^2(\mathbb{T})}.$$

Since $\xi'(\mu) = \phi \dfrac{dg_0}{d\mu} = -(1+\mu\phi)^{-1}\phi\xi(\mu)$ (according to (1.58)), it follows that

$$\begin{aligned} A_k'(\mu) \quad &= \quad -k\left\langle (1+\mu\phi)^{-k-1}\phi^k\xi(\mu), \xi(\mu)\right\rangle_{L^2(\mathbb{T})} - \left\langle (1+\mu\phi)^{-k-1}\phi^k\xi(\mu), \xi(\mu)\right\rangle_{L^2(\mathbb{T})} \\ &\quad - \left\langle (1+\mu\phi)^{-k}\phi^{k-1}\xi(\mu), (1+\mu\phi)^{-1}\phi\xi(\mu)\right\rangle_{L^2(\mathbb{T})}, \end{aligned}$$

and we thus arrive at the infinite-dimensional linear dynamical system

$$\begin{cases} A_k'(\mu) &= -(k+2)\,A_{k+1}(\mu)\,, \\ A_k(0) &= \left\langle \phi^{k-1}\xi_0, \xi_0 \right\rangle_{L^2} =: F(k-1)\,, \end{cases} \qquad k \in \mathbb{N}_+. \tag{1.73}$$

Introduce the matrix $\mathcal{M}$ whose powers are upper-diagonal with evident structure

$$\mathcal{M} = \begin{pmatrix} 0 & -3 & 0 & 0 & \dots \\ 0 & 0 & -4 & 0 & \dots \\ 0 & 0 & 0 & -5 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \mathcal{M}^2 = \begin{pmatrix} 0 & 0 & (-3)(-4) & 0 & \dots \\ 0 & 0 & 0 & (-4)(-5) & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \dots,$$

which makes the matrix exponential $e^{\mathcal{M}}$ easily computable. Then, due to such a structure, the system (1.73) is readily solvable, but of particular interest is the first component of the solution vector

$$A_1(\mu) \quad = \quad \sum_{k=1}^{\infty} \left[e^{\mathcal{M}\mu}\right]_{1,k} F(k-1) = \sum_{k=0}^{\infty} (-1)^k \, \frac{(k+2)!}{2} \frac{\mu^k}{k!} F(k)\,,$$

where the series converges for $|\mu| < 1$ since $F(k)$ is bounded by $\|\xi_0\|_{H^2}^2 = A_1(0) = F(0)$, as the Toeplitz operator ϕ is a contraction: $F(k)$ slowly decays to zero with k (see also plots and discussion at the end of Section 1.8). On the other hand, observe that, due to (1.59), $A_1(\mu) = -\dfrac{1}{2}\dfrac{dM_0^2}{d\mu}$ and thus

$$\frac{dM_0^2}{d\mu} = -\sum_{k=0}^{\infty} (-1)^k (k+1)(k+2)\,\mu^k F(k)\,. \tag{1.74}$$

Finally, termwise integration of (1.74) and use of (1.57) followed by rearrangement of terms furnish the results (1.71)-(1.72). □

Remark 1.5.2. *Note that when set $\psi \equiv 0$, $h \equiv 0$, it is seen that (1.74) can be obtained directly from (1.23), (1.59) which now reads*

$$\frac{dM_0^2}{d\mu} = -2Re\left\langle (1+\mu\phi)^{-3}\,\phi^2 P_+\left(\bar{b}f \vee 0\right), P_+\left(\bar{b}f \vee 0\right)\right\rangle_{L^2(\mathbb{T})}.$$

The result follows since a Neumann series (defining an analytic function for $|\mu| < 1$) is differentiable:

$$(1+\mu\phi)^{-1} = \sum_{k=0}^{\infty} (-1)^k \mu^k \phi^k \quad \Rightarrow \quad (1+\mu\phi)^{-3} = \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k (k+1)(k+2)\,\mu^k \phi^k.$$

We can also get some insight in behavior of $F(k)$ which lies in the heart of the series expansions (1.71)-(1.72) that will allow us to infer the bounds (1.68). First, we need the following

Lemma 1.5.1. *The sequence $\{F(k)\}_{k=0}^{\infty}$ is Abel summable*[2] *and it holds true that*

$$\lim_{\mu\to-1}\sum_{k=0}^{\infty}(-\mu)^k F(k)=e(0)<\infty. \tag{1.75}$$

Proof. Set $R_\mu(N):=\sum_{k=1}^{N}[F(k)-F(k-1)]\,k(-\mu)^k$ and apply summation by parts formula

$$R_\mu(N)=F(N)(N+1)(-\mu)^{N+1}+\mu F(0)-\sum_{k=1}^{N}F(k)\left((-\mu)^{k+1}(k+1)-(-\mu)^k k\right).$$

Passing to the limit and rearranging the terms, we obtain

$$\lim_{N\to\infty}R_\mu(N)=-\sum_{k=0}^{\infty}(-\mu)^{k+1}F(k)+(\mu+1)\sum_{k=1}^{\infty}(-\mu)^k kF(k),$$

and hence it follows from (1.72) that

$$e(\mu)=e(0)+(\mu+2)\sum_{k=0}^{\infty}(-\mu)^{k+1}F(k)+(\mu+1)\sum_{k=1}^{\infty}(-\mu)^{k+1}kF(k). \tag{1.76}$$

Combining the local integrability of $M_0^2(\mu)$, equivalent to (1.64), with the series expansion (1.71), we conclude that:

$$(\mu+1)\sum_{k=1}^{\infty}(-\mu)^k kF(k)\to 0 \text{ as } \mu\searrow-1.$$

Therefore, taking the limit $\mu\searrow-1$ in (1.76), the result (1.75) follows due to (1.62). $\square$

Now, without getting into detail of concrete spectral theory of Toeplitz operators, we only employ existence of a unitary transformation $U: H^2\to L^2_\lambda(\sigma)$ onto the spectral space where the Toeplitz operator is diagonal, meaning that its action simply becomes a multiplication by the spectral variable λ. Existence of such an isometry along with information on the spectrum of ϕ (Hartman-Wintner theorem, see Appendix), $\sigma=[0,1]$, and an assumption on the constant spectral density[3] $\rho_0>0$ make the following representation possible

$$F(k)=\int_0^1\lambda^k\left|(U\xi_0)(\lambda)\right|^2\rho_0 d\lambda \tag{1.77}$$

with $\int_0^1\left|(U\xi_0)(\lambda)\right|^2\rho_0 d\lambda=\|\xi_0\|^2_{H^2}$.

All the essential information on asymptotics (1.68) is contained in behavior of $(U\xi_0)(\lambda)$ near $\lambda=1$. Even though $(U\xi_0)(\lambda)$ can be computed since ξ_0 is a fixed function defined by (1.70) and the concrete spectral theory describes explicit action of the transformation U [8, 39], we avoid these details and proceed by deriving essential estimates invoking only rather intuitive arguments on the behavior of the resulting function $(U\xi_0)(\lambda)$.

Considering $-1<\mu<0$ in what follows, we, first of all, claim that the function $(U\xi_0)(\lambda)$ must necessarily

[2] By such summability we mean that $\sum_{k=0}^{\infty}\mu^kF(k)$ converges for all $|\mu|<1$ and the limit $\lim_{\mu\nearrow1}\sum_{k=0}^{\infty}\mu^kF(k)$ exists and is finite.

[3] Such an assumption is reasonable since the operator symbol χ_J is the simplest in a sense that it does not differ from one point to another in the region where it is non-zero and therefore the spectral mapping is anticipated to be uniform. Precise expression for the constant ρ_0 can be found in [8, 38].

decrease to zero as $\lambda \nearrow 1$. Indeed, even though L^2-behavior allows to have an integrable singularity at $\lambda = 1$, we note that even if regularity was assumed, that is $\lim_{\lambda \to 1} |(U\xi_0)(\lambda)|^2 = C$ for some $C > 0$, after summation of a geometric series, we would have

$$\frac{1}{\rho_0} \sum_{k=0}^{\infty} (-\mu)^k F(k) \geq C_0 \sum_{k=0}^{\infty} \int_{1-\delta}^{1} (-\mu\lambda)^k \, d\lambda = C_0 \int_{1-\delta}^{1} \frac{1}{1+\mu\lambda} d\lambda = \frac{C_0}{\mu} \log\left(\frac{1+\mu}{1+\mu-\mu\delta}\right)$$

for some $0 < C_0 \leq C$ and sufficiently small fixed $\delta > 0$. The right-hand side here grows arbitrary large as μ comes closer to -1 contradicting the boundedness prescribed by Lemma 1.5.1. Therefore, the decay to zero of $(U\xi_0)(\lambda)$ as $\lambda \nearrow 1$ is necessary.

Next, it is natural to proceed by checking if a very mild (meaning slower than any power) decay to zero can be reconciled with the previously obtained results. Namely, we consider $(U\xi_0)(\lambda)$ such that

$$|(U\xi_0)(\lambda)|^2 = \mathcal{O}\left(|\log(1-\lambda)|^{-l}\right) \quad \text{as} \quad \lambda \nearrow 1, \tag{1.78}$$

for $l > 1$. This entails the following result generalizing (1.68), see also Remarks 1.5.3, 1.5.4.

Proposition 1.5.4. *Under assumption (1.78) with $l > 1$, the solution blow-up and approximation rates near $\mu = -1$, respectively, are as follows*

$$M_0^2(\mu) = \mathcal{O}\left(\frac{1}{1+\mu} |\log(1+\mu)|^{-l}\right), \quad e(\mu) = \mathcal{O}\left(|\log(1+\mu)|^{-l+1}\right). \tag{1.79}$$

Proof. Choose a constant $0 < \lambda_0 < 1$ sufficiently close to 1 so that the asymptotic (1.78) is applicable. Therefore, we can write

$$\frac{1}{\rho_0} \sum_{k=0}^{\infty} (-\mu)^k F(k) = S_1 + S_2 + S_3$$

$$:= \int_0^{\lambda_0} \frac{1}{1+\mu\lambda} |(U\xi_0)(\lambda)|^2 \, d\lambda + \left(\int_{\lambda_0}^{1-\delta_0} + \int_{1-\delta_0}^{1}\right) \frac{1}{1+\mu\lambda} (-\log(1-\lambda))^{-l} \, d\lambda\,.$$

The first integral here is bounded regardless of the value of μ:

$$S_1 \leq \frac{1}{1+\mu\lambda_0} \int_0^1 |(U\xi_0)(\lambda)|^2 \, d\lambda = \frac{1}{(1+\mu\lambda_0)\rho_0} \|\xi_0\|_{H^2}^2\,.$$

To deal with S_3, we perform the change of variable $\beta = -\log(1-\lambda)$ and bound the factor $\dfrac{1}{\beta^l} \leq (-\log\delta_0)^{-l}$ to obtain

$$\int_{-\log\delta_0}^{\infty} \frac{1}{\beta^l} \frac{e^{-\beta}}{1+\mu-\mu e^{-\beta}} d\beta \leq \frac{1}{(-\mu)(-\log\delta_0)^l} \log\left(1 - \frac{\mu\delta_0}{1+\mu}\right) \leq \frac{\log 2}{(-\mu)|\log(1+\mu) - \log(-\mu)|^l}$$

providing we choose $\delta_0 \leq \dfrac{1+\mu}{(-\mu)}$. The quantity on the right is $\mathcal{O}\left(|\log(1+\mu)|^{-l}\right)$ in the vicinity of $\mu = -1$.

It remains to estimate S_2. The change of variable $\eta = 1-\lambda$ leads to

$$\begin{aligned} S_2 &= \int_{\delta_0}^{1-\lambda_0} \frac{\eta}{1+\mu-\mu\eta} \frac{1}{\eta\left(-\log\eta\right)^l} d\eta \le \left(\int_{\delta_0}^{1-\lambda_0} \frac{d\eta}{\eta\left(-\log\eta\right)^l}\right) \sup_{\eta\in[\delta_0,1-\lambda_0]} \left(\frac{\eta}{1+\mu-\mu\eta}\right) \\ &\le \frac{1}{l-1}\left(\frac{1}{\left|\log\delta_0\right|^{l-1}} - \frac{1}{\left|\log\left(1-\lambda_0\right)\right|^{l-1}}\right)\frac{1-\lambda_0}{1+\mu\lambda_0}. \end{aligned}$$

Therefore, we conclude that the choice (1.78) with $l>1$ does not contradict the finiteness imposed by Lemma 1.5.1 anymore and we move on to obtain the growth rate for $M_0^2(\mu)$ near $\mu=-1$. Recalling (1.71) and that $\sum_{k=0}^{\infty}\left(-\mu\lambda\right)^k(k+1) = \dfrac{1}{\left(1+\mu\lambda\right)^2}$, we now have

$$\frac{1}{\rho_0}\sum_{k=0}^{\infty}\left(-\mu\right)^k(k+1)F(k) = R_1+R_2+R_3+R_4$$

$$:= \int_0^{\lambda_0}\frac{1}{\left(1+\mu\lambda\right)^2}\left|\left(U\xi_0\right)(\lambda)\right|^2 d\lambda + \left(\int_{\lambda_0}^{1-\delta_1}+\int_{1-\delta_1}^{1-\delta_2}+\int_{1-\delta_2}^{1}\right)\frac{1}{\left(1+\mu\lambda\right)^2}\left(-\log\left(1-\lambda\right)\right)^{-l}d\lambda\,.$$

As before, we estimate

$$R_1 \le \frac{1}{\left(1+\mu\lambda_0\right)^2}\int_0^1\left|\left(U\xi_0\right)(\lambda)\right|^2 d\lambda = \frac{1}{\left(1+\mu\lambda_0\right)^2\rho_0}\left\|\xi_0\right\|_{H^2}^2,$$

whereas the rest is now split into 3 parts and we start with the last term and decide on proper size of δ_2 in

$$R_4 = \int_{1-\delta_2}^{1}\frac{1}{\left(1+\mu\lambda\right)^2}\frac{1}{\left|-\log\left(1-\lambda\right)\right|^l}d\lambda.$$

Again, under the change of variable $\beta = -\log\left(1-\lambda\right)$, this becomes

$$\begin{aligned} R_4 &= \frac{1}{\left(1+\mu\right)^2}\int_{-\log\delta_2}^{\infty}\frac{e^{-\beta}}{\beta^l}\frac{1}{\left(1-\frac{\mu}{1+\mu}e^{-\beta}\right)^2}d\beta \\ &= \frac{1}{\left(1+\mu\right)^2}\sum_{k=0}^{\infty}\left(\frac{\mu}{1+\mu}\right)^k(k+1)\int_{-\log\delta_2}^{\infty}\frac{e^{-(k+1)\beta}}{\beta^l}d\beta, \end{aligned}$$

where the series expansion is valid for $\delta_2 < \dfrac{1+\mu}{(-\mu)}$. The integral on the right is the incomplete gamma function (see, for instance, [2]) whose asymptotic expansion for large values of $\left(-\log\delta_2\right)$ can be easily obtained with integration by parts. In particular, at the leading order we have

$$\begin{aligned} \int_{-\log\delta_2}^{\infty}\frac{e^{-(k+1)\beta}}{\beta^l}d\beta &= (k+1)^{l-1}\int_{-(k+1)\log\delta_2}^{\infty}\frac{e^{-\beta}}{\beta^l}d\beta \\ &= (k+1)^{l-1}\,\delta_2^{k+1}\left(-(k+1)\log\delta_2\right)^{-l}\left[1+\mathcal{O}\left(\frac{1}{(k+1)\left|\log\delta_2\right|}\right)\right], \end{aligned}$$

and hence

$$R_4 = \frac{\delta_2}{\left(1+\mu\right)^2}\frac{1}{\left(-\log\delta_2\right)^l}\sum_{k=0}^{\infty}\left(\frac{\mu\delta_2}{1+\mu}\right)^k = \frac{\delta_2}{\left(1+\mu\right)^2}\frac{1}{\left(-\log\delta_2\right)^l}\frac{1}{1-\frac{\mu\delta_2}{1+\mu}}.$$

Fixing $\delta_2 = \dfrac{1}{2}\dfrac{1+\mu}{(-\mu)}$, we arrive at

$$R_4 = \frac{1}{(-\mu)(1+\mu)} \frac{1}{[-\log(1+\mu)+\log(-\mu)+\log 2]^l}.$$

To estimate R_2 and R_3, we use change of variable $\eta = 1-\lambda$. Similarly to S_2, we have

$$R_2 = \int_{\delta_1}^{1-\lambda_0} \frac{\eta}{(1+\mu-\mu\eta)^2} \frac{1}{\eta(-\log\eta)^l} d\eta \le \left(\int_{\delta_1}^{1-\lambda_0} \frac{d\eta}{\eta(-\log\eta)^l}\right) \sup_{\eta\in[\delta_1, 1-\lambda_0]} \left(\frac{\eta}{[1+\mu-\mu\eta]^2}\right),$$

however, now under the supremum sign, instead of a monotonic function, we have an expression that attains a maximum value $\dfrac{1}{4(-\mu)(1+\mu)}$ if $\delta_1 < \dfrac{1+\mu}{(-\mu)}$ which lacks the smallness we obtained in R_4. Therefore, to remedy the situation, we require $\delta_1 > \dfrac{1+\mu}{(-\mu)}$ and obtain

$$R_2 \le \frac{1}{l-1}\left(\frac{1}{|\log\delta_1|^{l-1}} - \frac{1}{|\log(1-\lambda_0)|^{l-1}}\right) \frac{\delta_1}{(1+\mu-\mu\delta_1)^2} = \mathcal{O}\left(\frac{1}{1+\mu}|\log(1+\mu)|^{-\gamma}\right)$$

near $\mu=-1$, if we fix $\delta_1 = \dfrac{1+\mu}{(-\mu)}(1+[-\log(1+\mu)]^\gamma)$ for arbitrary $\gamma > 0$.
The last integral R_3 is to bridge the gap between the two neighborhoods of $\lambda = 1$:

$$R_3 = \int_{\delta_2}^{\delta_1} \frac{1}{(1+\mu-\mu\eta)^2}\frac{1}{(-\log\eta)^l} d\eta \le \frac{1}{(-\log\delta_1)^l}\left(\frac{1}{1+\mu-\mu\delta_2} - \frac{1}{1+\mu-\mu\delta_1}\right)$$

and hence, using the fact that $\log(-\log(1+\mu)) = o(-\log(1+\mu))$, we deduce that near $\mu=-1$

$$R_3 = \mathcal{O}\left(\frac{1}{1+\mu}|\log(1+\mu)|^{-l}\right).$$

Now that all the integral terms are estimated, choice of the parameter $\gamma = l$ in δ_1 leads to the first estimate in (1.79) whereas integration of (1.57) recovers the second one. □

Remark 1.5.3. *The case $l=2$ gives exactly the expressions in (1.68). The assumed behavior* (1.78) *of $(U\xi_0)(\lambda)$ is analogous (with direct correspondence in the case $\psi \equiv 0$, $h \equiv 0$) to the conclusion of [8, Prop. 4.1] which was used to generate further estimates therein, and the case $l=3$ is related to improved estimates given in [8, Cor. 4.6] under assumption of even higher regularity of boundary data (roughly speaking, integrability of second derivatives). It is noteworthy that the choice $l=1$ yields non-integrable behavior of $M_0^2(\mu)$ contradicting Proposition 1.5.2, and therefore was eliminated in the formulation. This is not due to the fact that the method of estimation of the S_2 integral fails, but because of non-integrability near $\mu=-1$ of the overall bound. The R_4 term has been computed asymptotically sharply though it could be made even smaller by shrinking the neighborhood δ_2. Indeed, instead of the $\dfrac{1}{2}$ factor in δ_2, we could have put $\dfrac{1}{1+[-\log(1+\mu)]^\beta}$ for any $\beta \ge 0$ similarly to what was done in the R_2 term which allowed a multiplier with arbitrary logarithmical smallness regulated by the parameter γ. This, however, would not reduce the overall blow-up because of the stiff bridging term R_3. Even though the estimate for R_3 is rough, we do not expect improvement by an order of magnitude because the logarithmic factor of the integrand* *picks up $(1+\mu)$ as a major multiplier near $\eta=\delta_1$ which makes any choice of $\gamma \geq l$ and $\beta \geq 0$ useless in attempt to improve the smallness factor in the blow-up of $M_0^2(\mu)$.*

Remark 1.5.4. *Generally, we note that the appearance of the $\log(1+\mu)$ factors in the bounds is not accident, but intrinsically encoded in the connection between $e(\mu)$ and $M_0^2(\mu)$ since (1.57) can be rewritten as $e'(\mu) = -\dfrac{dM_0^2}{d[\log(1+\mu)]}$ which also explains the choice of (1.78).*

We would like to point out again that even though our reasoning was meant to provide an intuitive explanation of the estimates (1.68), more rigourous proofs can be found in [8] where an elegant connection of the bounds with regularity of given boundary data is established by elaborating concrete spectral theory results [39] into formulation of a certain integral transformation followed by application of L^1-theory of Fourier transforms (Riemann-Lebesgue lemma). Also, one can take an alternative viewpoint based on the results of [38]. In that case, the unitary transformation U diagonalizing the Toeplitz operator ϕ acts on Fourier coefficients $\{\eta_n\}_{n=0}^{\infty} \in l^2(\mathbb{N}_0)$ of a given $\xi_0 \in H^2$ as

$$(U\xi_0)(\lambda) = \sum_{n=0}^{\infty} \eta_n \psi_n(\lambda), \tag{1.80}$$

where the orthonormal sequence of $L^2(0,1)$ functions $\psi_n(\lambda)$ are explicitly defined in terms of the Meixner-Pollaczek polynomials of order $1/2$ [36]:

$$\psi_n(\lambda) := e^{a\beta}\left(1+e^{-2\pi\beta}\right)^{1/2} P_n^{(1/2)}(\beta, a), \qquad \beta := -\frac{1}{2\pi}\log\left(\frac{1}{\lambda}-1\right)$$

providing $I=\left(e^{-ia}, e^{ia}\right)$, $a \in (0,\pi)$, an assumption that does not reduce the generality if the original sets I and J are two disjoint arcs.

A recurrence formula for the Meixner-Pollaczek polynomials follows from that for the Pollaczek polynomials [42]:

$$P_n^{(1/2)}(\beta) = \frac{1}{n}\left(2\beta\sin a - (2n-1)\cos a\right) P_{n-1}^{(1/2)}(\beta) - \frac{n-1}{n} P_{n-2}^{(1/2)}(\beta), \tag{1.81}$$

$$P_{-1}^{(1/2)}(\beta) = 0, \quad P_0^{(1/2)}(\beta) = 1,$$

which allows to generate all the coefficients $k_m^{(n)}$ in $P_n^{(1/2)}(\beta) = \sum_{m=0}^{n} k_m^{(n)} \beta^m$, for instance,

$$k_n^{(n)} = \frac{(2\sin a)^n}{n!}, \quad k_{n-1}^{(n)} = -n\cos a \frac{(2\sin a)^{n-1}}{(n-1)!},$$

$$k_{n-2}^{(n)} = \frac{1}{6}\left[3n(n-1)\cos^2 a - (2n-1)\sin^2 a\right] \frac{(2\sin a)^{n-2}}{(n-2)!}.$$

Rearranging the terms in (1.80), we can write (suppressing the first two factors for the sake of compactness)

$$\begin{aligned}(U\xi_0)(\lambda) &\propto \sum_{n=0}^{\infty}\eta_n\sum_{m=0}^{n}k_m^{(n)}\beta^m=\sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}\eta_n k_m^{(n)}\right)\beta^m\\ &= \sum_{m=0}^{\infty}\left(\eta_m k_m^{(m)}+\eta_{m+1}k_m^{(m+1)}+\dots\right)\beta^m. \end{aligned}\tag{1.82}$$

It would be interesting to see, in such a representation, what decay assumptions on the Fourier coefficients η_n are consistent with (1.78), and thus (1.79), with $1<l<2$ in which case there is no violation of integrability of $M_0^2(\mu)$ and less regularity assumptions (namely, milder than decay of $n\eta_n$ to zero as $n\to\infty$) are expected than those related with integrability of the first derivative of boundary data.

Note that, because of the Taylor series of the exponential function, we have

$$\begin{aligned}\left|\sum_{m=0}^{\infty}\left(\eta_m k_m^{(m)}\right)\beta^m\right| &\le \left(\sup_{m\in\mathbb{N}_0}|\eta_m|\right)\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\sin a}{\pi}\right)^m\left|\log\left(\frac{1}{\lambda}-1\right)\right|^m\\ &= \left(\sup_{m\in\mathbb{N}_0}|\eta_m|\right)\begin{cases}\left(\dfrac{1}{\lambda}-1\right)^{\dfrac{\sin a}{\pi}}, & 0<\lambda<\dfrac{1}{2},\\ \left(\dfrac{1}{\lambda}-1\right)^{-\dfrac{\sin a}{\pi}}, & \dfrac{1}{2}\le\lambda<1,\end{cases}\end{aligned}$$

and thus the very first term already adds to the singular behavior of (1.80) near $\lambda=1$ (unless additional assumptions on alternation of sign of η_m are made) instead of revealing any decay to zero. This suggests that terms in the brackets of (1.82) should not be estimated separately: the other terms contribute equally to $(U\xi_0)(\lambda)$ though their expressions are much more cumbersome for straightforward analysis.

An alternative way might be to work in direction of obtaining estimates of (1.71)-(1.72) near $\mu=-1$ in terms of η_m from

$$\int_0^1\frac{1}{1+\mu\lambda}\left|(U\xi_0)(\lambda)\right|^2d\lambda \quad\text{and}\quad \int_0^1\frac{1}{(1+\mu\lambda)^2}\left|(U\xi_0)(\lambda)\right|^2d\lambda$$

directly without deducing behavior of $(U\xi_0)(\lambda)$ in vicinity of $\lambda=1$, but using explicit form of the unitary transformation (1.80). To take advantage of it, one can potentially expand integrand factors $\dfrac{1}{1+\mu\lambda}$ in terms of β and iteratively employ the recurrence formula (1.81) rewritten as

$$\beta P_n^{(1/2)}(\beta,a)=\frac{n+1}{2\sin a}P_{n+1}^{(1/2)}(\beta,a)+\frac{(2n+1)\cot a}{2}P_n^{(1/2)}(\beta,a)+\frac{n}{2\sin a}P_{n-1}^{(1/2)}(\beta,a)$$

followed by application of orthonormality. Note that such a strategy (but based on expansion of λ in terms of β) along with the fact that $U^{-1}\psi_n(\lambda)=z^n$ might also be used to see how the Toeplitz operator ϕ acts on Fourier coefficients of a function.

1.6 Companion problem

At this moment, it is time to point out a link with another bounded extremal problem which relies on the observation that formal substitution of $\mu = 0$ in (1.53) implies that

$$\tilde{g}_0 = \psi + bP_+ \left(\bar{b}\left(f \vee h\right)\right) \tag{1.83}$$

is an explicit solution for the problem with the particular constraint

$$M = M_0\left(0\right) = \left\|\psi + bP_+ \left(\bar{b}\left(f \vee h\right)\right) - h\right\|_{L^2(J)}.$$

Recalling that $bP_+\bar{b}$ is a projector onto bH^2 (see Section 1.2), we note that, geometrically, the solution (1.83) is simply a realization of projection of $f \vee h \in L^2\left(\mathbb{T}\right)$ onto $\mathcal{C}^{\psi,b}_{M,h}$. Now, exploiting the arbitrariness of choice of interpolant ψ (Remark 1.4.1), we can change our viewpoint and look for $\psi \in H^2$ meeting pointwise constraints (1.13) such that $\psi + bP_+ \left(\bar{b}\left(f \vee h\right)\right) - h$ is sufficiently close to the constant[4] $M/\sqrt{|J|}$ in $L^2\left(J\right)$ yet remaining L^2-bounded on I. In other words, given arbitrary $\psi_0 \in H^2$ satisfying the pointwise interpolation conditions (1.13) (take, for instance, (1.46)), we represent $\psi = \psi_0 + b\Psi$ and thus search for an approximant $\Psi \in H^2$ to $\bar{b}\left(h - \psi_0 + M/\sqrt{|J|}\right) - P_+\left(\bar{b}\left(f \vee h\right)\right) \in L^2\left(J\right)$ such that $\left\|\Psi\right\|_{L^2(I)} = K$ for arbitrary $K \in (0, \infty)$. We thus reduce the original problem to an associated approximation problem on J for which all known data are now prescribed on J alone. Since the constraint on I is especially simple (role of ψ and h play identically zero functions), such a companion problem has a computational advantage over the original one as, due to the form of solution (1.23), it requires integration only over a subset of $\mathbb{T}$ (see (1.107)).

To be more precise, let Ψ_0 be a solution to the companion problem such that

$$\left\|\psi_0 + b\Psi_0 + bP_+ \left(\bar{b}\left(f \vee h\right)\right) - h\right\|^2_{L^2(J)} = M^2 + \delta_{M^2},$$

where δ_{M^2} measures accuracy of the solution of the companion problem. Then, solution to the original problem should be sought as a series expansion near (1.83) with respect to δ_{M^2} as a small parameter

$$\tilde{g}_0 = \psi_0 - bP_+\left(\bar{b}\psi_0\right) + bP_+ \left(\bar{b}\left(f \vee h\right)\right) + b\left.\frac{dg_0}{d\mu}\right|_{\mu=0} \left.\frac{d\mu}{dM_0^2}\right|_{M_0^2=M^2} \delta_{M^2} + \dots, \tag{1.84}$$

and further the relations (1.58)-(1.59) followed by $\left.\frac{d\mu}{dM_0^2}\right|_{M_0^2=M^2} = \left.\left(\frac{dM_0^2}{d\mu}\right)^{-1}\right|_{\mu=0}$ should be employed (here g_0 is as in (1.23)). Recalling Section 1.2, we note that the first two terms realize a projection of ψ_0 onto $\left(bH^2\right)^{\perp_{H^2}}$ which will be simply ψ_0 if (1.46) was used as the arbitrary interpolant (see Remark 1.4.1).

If the companion problem was solved with good accuracy so that δ_{M^2} is small, linear order approximation in δ_{M^2} may be sufficient to recover the solution of the original problem. However, this connection between solution of

[4] Alternatively, one can take any $L^2\left(J\right)$ function that has norm M.

two problems is valid for arbitrary values of δ_{M^2} if one considers infinite series in δ_{M^2}. This can be formalized with use of the Faà di Bruno formula which provides explicit form of the Taylor expansion for the function composition $g_0\left(\mu\left(M_0^2\right)\right)$ in terms of the derivatives $\left.\frac{d^k g_0}{(d\mu)^k}\right|_{\mu=0}$ and $\left.\frac{d^k \mu}{\left(dM_0^2\right)^k}\right|_{M_0^2=M^2}$ for any $k \in \mathbb{N}_+$. Applying the product rule and expression (1.58) successively it can be shown that, after collection of terms at each differentiation, we have

$$\frac{d^k g_0}{(d\mu)^k} = (-1)^k\, k!\,(1+\mu\phi)^{-k}\,\phi^k\tilde{\xi} \quad \Rightarrow \quad \left.\frac{d^k g_0}{(d\mu)^k}\right|_{\mu=0} = (-1)^k\, k!\phi^k\tilde{\xi}_0,$$

where

$$\tilde{\xi} := P_+\left(0 \vee \left(g_0 + \bar{b}\left(\psi_0 - h\right) + \Psi_0\right)\right), \quad \tilde{\xi}_0 := \phi\left(P_+\left(\bar{b}\left(f \vee h\right)\right) - P_+\left(\bar{b}\psi_0\right) - \Psi_0\right).$$

As far as computation of derivates of $\frac{d\mu}{dM_0^2}$ is concerned, complexity of the expressions grows and precise pattern seem to be hard to find especially since implicit differentiation has to be repeated every time resulting in successive appearance of extra factor $\frac{d\mu}{dM_0^2}$. Even though in practice one may look at the truncated Taylor expansion $M_0^2(\mu)$ and, since derivatives $\frac{dM_0^2}{d\mu}$ are readily computable, use reversion of the series to obtain power series expansion of μ in terms of M_0^2 (for reversion of series coefficient formula, see [34]) or, alternatively, employ the Lagrange inversion theorem that yields the inverse function $\mu\left(M_0^2\right)$ as an infinite series, in the latter case we would have to decide at which term the both series should be truncated so that to preserve desired accuracy at given order of δ_{M^2}. For small δ_{M^2}, only few terms are needed to give quite accurate connection between solution of the original and companion problems. Those can be precomputed manually or using computer algebra systems once and such calculations need not be repeated iteratively.

1.7 Stability results

The issue to be discussed here is linear stability of the solution (1.18) with respect to all physical components that the expression (1.23) involves explicitly and implicitly. In practice, functions f, h are typically obtained by interpolating discrete boundary data and hence may vary depending on interpolation method, measurement positions $\{z_j\}_{j=1}^N$ are usually known with a small error and pointwise data $\{\omega_j\}_{j=1}^N$ are necessarily subject to a certain noise. Therefore, we assume that boundary data f, h are slightly perturbed by $\delta_f \in L^2(I)$, $\delta_h \in L^2(J)$ and internal data $\{\omega_j\}_{j=1}^N$ with measurement positions $\{z_j\}_{j=1}^N$ by complex vectors $\boldsymbol{\delta_\omega}$, $\boldsymbol{\delta_z} \in \mathbb{C}^N$, respectively. Varying one of the quantities while the rest are kept fixed, we are going to estimate separately the linear effects of such perturbations on the solution $\tilde{g}_0 = \psi + bg_0$ to (1.18), denoting the induced deviations as $\delta_{\tilde{g}}$.

Proposition 1.7.1. *For $\mu > -1$, $f \in L^2(I) \setminus \left.\mathcal{A}^{\psi,b}\right|_I$, $h \in L^2(J)$, and small enough data perturbations $\delta_f \in L^2(I)$, $\delta_h \in L^2(J)$, $\boldsymbol{\delta_\omega}$, $\boldsymbol{\delta_z} \in \mathbb{C}^N$, the following estimates hold:*

(1) $\|\delta_{\tilde{g}}\|_{H^2} \leq m_1\left(1 + \frac{m_1 M^2}{m_0\|\xi\|_{H^2}^2}\right)\|\delta_f\|_{L^2(I)}$,

(2) $\|\delta_{\tilde{g}}\|_{H^2} \leq \left[(1 + m_1(1+\mu))\left(1 + \frac{m_1 M^2}{m_0\|\xi\|_{H^2}^2}\right) - 1\right]\|\delta_h\|_{L^2(J)}$,

(3) $\|\delta_{\tilde{g}}\|_{H^2} \leq (1+|\mu|\, m_1)\left(1+\dfrac{m_1 M^2}{m_0\|\xi\|^2_{H^2}}\right)\max\limits_{j=1,\dots,N}\left\|\prod\limits_{\substack{k=1\\k\neq j}}^{N}\dfrac{z-z_k}{z_j-z_k}\right\|_{H^2}\|\boldsymbol{\delta_\omega}\|_{l^1}$,

(4) $\|\delta_{\tilde{g}}\|_{H^2} \leq \left(1+\dfrac{m_1 M^2}{m_0\|\xi\|^2_{H^2}}\right)\left(C^{(1)}_\mu\|\delta_b\|_{H^\infty}+C^{(2)}_\mu\|\delta_\psi\|_{H^2}\right)$,

where

$$\xi := P_+\left(0\vee\left(g_0+\bar{b}\,(\psi-h)\right)\right),\;\; m_0 := \min\left\{(1+\mu)^{-1},1\right\},\;\; m_1 := \max\left\{(1+\mu)^{-1},1\right\}, \tag{1.85}$$

$C^{(1)}_\mu := m_1\left(\|f\vee h\|_{L^2(\mathbb{T})}+|\mu|\,\|h-\psi\|_{L^2(J)}\right)$, $\quad C^{(2)}_\mu := 1+|\mu|\,m_1,$ $\qquad$ *and*

$\|\delta_b\|_{H^\infty} \leq 2\max\limits_{j=1,\dots,N}\left\|(z-z_j)^{-1}\right\|_{H^\infty}\|\boldsymbol{\delta_z}\|_{l^1}$,

$\|\delta_\psi\|_{H^2} \leq 2\max\limits_{j=1,\dots,N}|\omega_j|\max\limits_{j=1,\dots,N}\left\|\prod\limits_{\substack{m=1\\m\neq j}}^{N}(z-z_m)\right\|_{H^2}\times$

$\max\limits_{j=1,\dots,N}\sum\limits_{\substack{k=1\\k\neq j}}^{N}|z_j-z_k|^{-1}\left(\min\limits_{j=1,\dots,N}\prod\limits_{\substack{k=1\\k\neq j}}^{N}|z_j-z_k|\right)^{-1}\|\boldsymbol{\delta_z}\|_{l^1}$,

Proof. When the quantities entering the solution (1.23) vary, the overall variation of the solution δ_g will consist of parts entering the solution formula explicitly δ_{g_0} as well as those coming from the change of the norm of g_0 on J which leads to readjustment of the Lagrange parameter δ_μ so that the quantity $M_0^2(\mu) = \|\psi+bg_0(\mu)-h\|^2_{L^2(J)}$ be equal to the same given constraint M^2. For the sake of brevity, we are going to use the notations ξ, m_0 and m_1 introduced in (1.85) to denote certain quantities entering common estimates. The spectral bounds (1.25) for $\mu > -1$ imply

$$\sigma(1+\mu\phi) \geq \min\{1+\mu,1\}, \quad \sigma(1+\mu\phi)\leq\max\{1+\mu,1\}$$

$$\Rightarrow \quad \left\|(1+\mu\phi)^{-1}\right\| \leq \max\left\{(1+\mu)^{-1},1\right\}, \quad \left\|(1+\mu\phi)^{-1}\right\| \geq \min\left\{(1+\mu)^{-1},1\right\},$$

and so, in particular,

$$\mathrm{Re}\left\langle(1+\mu\phi)^{-1}\xi,\xi\right\rangle_{L^2(\mathbb{T})} \geq m_0\|\xi\|^2_{H^2}.$$

Then, the connection between δ_{M^2} denoting the change of $M_0^2(\mu)$ and δ_μ can be established based on the strict monotonicity (1.59) of $M_0(\mu)$ which allows the following estimate by inversion

$$\delta_\mu = \frac{\delta_{M^2}}{\left(M_0^2(\mu)\right)'} = -\frac{\delta_{M^2}}{2\mathrm{Re}\left\langle(1+\mu\phi)^{-1}\xi,\xi\right\rangle_{L^2(\mathbb{T})}} \quad\Rightarrow\quad |\delta_\mu| \leq \frac{|\delta_{M^2}|}{2m_0\|\xi\|^2_{H^2}}. \tag{1.86}$$

Note that the bound in the right-hand side is finite due to the fact that $\|\xi\|_{H^2} > 0$ which holds unless $M_0(\mu)=0$, the situation that was initially ruled out by Corollary 1.3.1. Discussion on *a priori* estimate of $\|\xi\|_{H^2}$ will be given in Remark 1.7.1.

Following this strategy, we embark on consecutive proof of the results *(1)-(4)*.

Result (1):

This is the simplest case, the variation of $M_0^2(\mu)$ is induced only by change of g_0. Namely,

$$\delta_{M^2} = 2\mathrm{Re}\left\langle \psi + bg_0(\mu) - h, b\delta_{g_0}(\mu)\right\rangle_{L^2(J)}, \tag{1.87}$$

where

$$\delta_{g_0} = (1+\mu\phi)^{-1} P_+\left(\bar{b}\delta_f \vee 0\right). \tag{1.88}$$

Application of the Cauchy-Schwarz inequality to (1.87) yields

$$|\delta_{M^2}| \quad \leq \quad 2M_0(\mu)\left\|(1+\mu\phi)^{-1}\right\| \left\|P_+\left(\bar{b}\delta_f \vee 0\right)\right\|_{L^2(\mathbb{T})} \leq 2M_0(\mu)\, m_1 \left\|\delta_f\right\|_{L^2(I)}.$$

and hence, by (1.86),

$$|\delta_\mu| \leq \frac{m_1 M_0(\mu)}{m_0\left\|\xi\right\|^2_{H^2}}\left\|\delta_f\right\|_{L^2(I)}.$$

Now since $\delta_{\tilde{g}} = b\delta_g$, due to (1.58), we have

$$\delta_{\tilde{g}} = b\delta_{g_0} - b(1+\mu\phi)^{-1} P_+\left(0 \vee \left(g_0 + \bar{b}(\psi - h)\right)\right)\delta_\mu, \tag{1.89}$$

from where we deduce the inequality *(1)*.

Result (2):

This is totally analogous to the previous result except for now we have

$$\delta_{M^2} = 2\mathrm{Re}\left\langle \psi + bg_0(\mu) - h, b\delta_{g_0}(\mu) - \delta_h\right\rangle_{L^2(J)} \tag{1.90}$$

with

$$\delta_{g_0} = (1+\mu\phi)^{-1} P_+\left(0 \vee (1+\mu)\,\bar{b}\delta_h\right). \tag{1.91}$$

Therefore,

$$|\delta_{M^2}| \leq 2M_0(\mu)\left[1 + (1+\mu)\, m_1\right]\left\|\delta_h\right\|_{L^2(J)} \quad \Rightarrow \quad |\delta_\mu| \leq \frac{M_0(\mu)\left[1 + (1+\mu)\, m_1\right]}{m_0\left\|\xi\right\|^2_{H^2}}\left\|\delta_h\right\|_{L^2(J)}.$$

Feeding this in the relation (1.89), which still holds in this case, gives

$$\left\|\delta_{\tilde{g}}\right\|_{H^2} \leq m_1\left(1 + \mu + \frac{\left[1 + (1+\mu)\, m_1\right] M^2}{m_0\left\|\xi\right\|^2_{H^2}}\right)\left\|\delta_h\right\|_{L^2(J)},$$

that is exactly a rewording of estimate *(2)*.

Result (3):

The estimates *(3)* and *(4)* explore sensitivity of solution to measurement noise which any experimental data are prone to. In both cases proofs are similar to those of *(1)-(2)* with only few new ingredients.

In case of *(3)*, a perturbed data vector $\boldsymbol{\delta_\omega} \in \mathbb{C}^N$ affects the solution $\tilde{g}_0$ by means of the induced variation of ψ

that we will denote by $\delta_\psi \in H^2(\mathbb{D})$.

If ψ is given by (1.46), its perturbation can be estimated as

$$\|\delta_\psi\|_{H^2} \leq \max_{k=1,\dots,N} \|\mathcal{K}(z_k,\cdot)\|_{H^2} \|S\|_1 \|\boldsymbol{\delta_\omega}\|_{l^1}, \tag{1.92}$$

where $\|\boldsymbol{\delta_\omega}\|_{l^1} := \sum_{k=1}^N |(\delta_\omega)_k|$, $\|S\|_1 := \max_{j=1,\dots,N} \sum_{k=1}^N |S_{kj}|$ with S as defined in (1.47). However, to get more explicit result with respect to data positions $\{z_j\}_{j=1}^N$ (which will be more relevant in case *(4)*) avoiding reference to (1.47), we employ polynomial interpolation in Lagrange form

$$\psi = \sum_{j=1}^N \omega_j \prod_{\substack{k=1\\k\neq j}}^N \frac{z-z_k}{z_j-z_k}, \tag{1.93}$$

in which case we have

$$\|\delta_\psi\|_{H^2} \leq \max_{j=1,\dots,N} \left\| \prod_{\substack{k=1\\k\neq j}}^N \frac{z-z_k}{z_j-z_k} \right\|_{H^2} \|\boldsymbol{\delta_\omega}\|_{l^1}. \tag{1.94}$$

Nevertheless, we note that the choice of interpolant (1.93) is not good for practical usage (making way for the barycentric interpolation formula, see [14]), but done only for the sake of analysis (again recall that, by Lemma 1.4.1, the final solution $\tilde{g}_0$ does not depend on a particular choice of the interpolant). In particular, we see that closedness of interpolation points amplifies the bound in the right-hand side which corresponds to ill-conditioning of the matrix $\mathcal{K}(z_k, z_j)$ for the choice of interpolant (1.46).

From this point on, we follow the same steps as in case *(2)* with (1.90)-(1.91) replaced by

$$\delta_{M^2} = 2\mathrm{Re}\left\langle \psi + bg_0(\mu) - h, \delta_{\tilde{g}_0}(\mu)\right\rangle_{L^2(J)}, \tag{1.95}$$

$$\delta_{\tilde{g}_0} = \delta_\psi - \mu(1+\mu\phi)^{-1} P_+\left(0 \vee \bar{b}\delta_\psi\right), \tag{1.96}$$

where the latter variation is estimated from (1.53). Then, we have

$$|\delta_{M^2}| \leq 2M_0(\mu)(1+|\mu| m_1)\|\delta_\psi\|_{L^2(J)} \quad \Rightarrow \quad |\delta_\mu| \leq \frac{M_0(\mu)(1+|\mu| m_1)}{m_0 \|\xi\|_{H^2}^2} \|\delta_\psi\|_{L^2(J)}. \tag{1.97}$$

Now

$$\delta_{\tilde{g}} = \delta_{\tilde{g}_0} - b(1+\mu\phi)^{-1} P_+\left(0 \vee \left(g_0 + \bar{b}(\psi - h)\right)\right)\delta_\mu, \tag{1.98}$$

and the resulting estimate *(3)* follows using (1.96)-(1.97) and recalling (1.94).

Result (4):

For a perturbation vector of positions $\boldsymbol{\delta_z} \in \mathbb{C}^N$, the respective deviation of the interpolant (1.93) is given by

$$\delta_\psi = \sum_{j=1}^N \omega_j \sum_{\substack{k=1\\k\neq j}}^N \left(\prod_{\substack{m=1\\m\neq k,j}}^N \frac{z-z_m}{z_j-z_m} \right) \frac{(z-z_j)(\delta_z)_k - (z-z_k)(\delta_z)_j}{(z_j-z_k)^2}, \tag{1.99}$$

and can be bounded, for instance, as

$$\|\delta_\psi\|_{H^2} \leq 2\omega_0 \max_{j=1,\dots,N} \left\| \prod_{\substack{m=1\\m\neq j}}^{N} (z - z_m) \right\|_{H^2} \frac{\max\limits_{j=1,\dots,N} \sum_{\substack{k=1\\k\neq j}}^{N} |z_j - z_k|^{-1}}{\min\limits_{j=1,\dots,N} \prod_{\substack{k=1\\k\neq j}}^{N} |z_j - z_k|} \|\boldsymbol{\delta_z}\|_{l^1}, \tag{1.100}$$

where $\omega_0 := \max\limits_{j=1,\dots,N} |\omega_j|$. However, more compact but even rougher bounds can be obtained in terms of d_0^{-N}, where $d_0 := \min\limits_{\substack{j,k=1,\dots,N\\j\neq k}} |z_j - z_k|$, which are undesirable for large number of points that are not uniformly spaced.

This case is the most tedious one since now, in addition to ψ, the Blaschke products undergo the variation

$$\delta_b = \sum_{j=1}^{N} \left(\prod_{\substack{m=1\\m\neq j}}^{N} \frac{z - z_m}{1 - \bar{z}_m z} \right) \frac{z (z - z_j) (\delta_{\bar{z}})_j - (1 - \bar{z}_j z) (\delta_z)_j}{(1 - \bar{z}_j z)^2}, \tag{1.101}$$

which can be estimated as

$$\begin{aligned} \|\delta_b\|_{H^\infty} &\leq \max_{j=1,\dots,N} \left(\left\| \frac{z (z - z_j)}{(1 - \bar{z}_j z)^2} \right\|_{H^\infty} + \left\| (1 - \bar{z}_j z)^{-1} \right\|_{H^\infty} \right) \|\boldsymbol{\delta_z}\|_{l^1} \\ &= 2 \max_{j=1,\dots,N} \left\| (z - z_j)^{-1} \right\|_{H^\infty} \|\boldsymbol{\delta_z}\|_{l^1}. \end{aligned} \tag{1.102}$$

The rest of the computations is most similar to those in case *(3)* but slightly more general. Namely, (1.95) and (1.98) hold with

$$\begin{aligned} \delta_{\tilde{g}_0} &= \delta_\psi + \delta_b (1 + \mu\phi)^{-1} \left[P_+ \left(\bar{b} (f \vee h) \right) + \mu P_+ \left(0 \vee \bar{b} (h - \psi) \right) \right] \\ &\quad + b (1 + \mu\phi)^{-1} \left[P_+ (\delta_{\bar{b}} (f \vee h)) + \mu P_+ (0 \vee \delta_{\bar{b}} (h - \psi)) \right] - \mu b (1 + \mu\phi)^{-1} P_+ \left(0 \vee \bar{b} \delta_\psi \right) \end{aligned}$$

estimated from (1.53). Therefore,

$$\|\delta_{\tilde{g}}\|_{H^2} \leq m_1 \left(1 + \frac{m_1 M^2}{m_0 \|\xi\|_{H^2}^2} \right) \|\delta_{\tilde{g}_0}\|_{H^2},$$

$$\|\delta_{\tilde{g}_0}\|_{H^2} \leq (1 + |\mu| m_1) \|\delta_\psi\|_{H^2} + m_1 \left(\|f \vee h\|_{L^2(\mathbb{T})} + |\mu| \|h - \psi\|_{L^2(J)} \right) \|\delta_b\|_{H^\infty},$$

and the final estimate *(4)* follows. □

Remark 1.7.1. *The quantity ξ introduced in (1.85) enters the results (1)-(4) and should be bounded away from zero. This fact, however, follows from Proposition 1.2.2 and Corollary 1.3.1. Moreover, the norm of ξ can be a priori estimated as*

$$\|\xi\|_{H^2} \geq \frac{1}{|\mu|} \left(M - \left\| \psi - h + bP_+ \left(\bar{b} (f \vee h) \right) \right\|_{L^2(J)} \right) \tag{1.103}$$

by applying the triangle inequality for $L^2(J)$ norm of the quantity

$$\psi + bg_0 - h = \psi - h + bP_+ \left(\bar{b} (f \vee h) \right) + \mu b P_+ \left(0 \vee \left(\bar{b} (h - \psi) - g_0 \right) \right),$$

which is a consequence of (1.27). Of course, the estimate (1.103) is useful only under assumption

$$\left\|\psi - h + bP_+ \left(\bar{b}\left(f \vee h\right)\right)\right\|_{L^2(J)} < M, \tag{1.104}$$

but we do not include it in formulation of the Proposition, since this inequality can be achieved without imposing any restriction on given boundary data f and h or increasing the bound M: since, according to Lemma 1.4.1, choice of ψ does not affect solution $\tilde{g}_0$ whose stability we are investigating, one can consider another instance of bounded extremal problem, now formulated for $\psi \in H^2(\mathbb{D})$ meeting pointwise constraints (1.13) and approximating $h - bP_+\left(\bar{b}\left(f \vee h\right)\right) \in L^2(J)$ on J sufficiently closely (with precision M) with a finite bound on I without any additional information (meaning that for such a problem "I" $= J$, "h" $= 0$). To be more precise, given arbitrary $\psi_0 \in H^2(\mathbb{D})$ satisfying pointwise interpolation conditions (1.13) (for instance, one can use (1.46)), we represent $\psi = \psi_0 + b\Psi$ and thus search for approximant $\Psi \in H^2(\mathbb{D})$ to "f" $= \bar{b}\left(h - \psi_0\right) - P_+\left(\bar{b}\left(f \vee h\right)\right) \in L^2(J)$ such that $\|\Psi\|_{L^2(I)} = \tilde{M}$ for arbitrary $\tilde{M} \in (0, \infty)$. We also note that in the case of reduction to the previously considered problem with no pointwise data imposed ([5], [7]), i.e. when $\psi \equiv 0$, $b \equiv 1$, one does not have flexibility of varying the interpolant. However, the stability estimates still persist in the region of interest (that is, for $-1 < \mu < 0$) since the condition (1.104) is fulfilled as long as $\mu < 0$ due to (1.23) evaluated at $\mu = 0$ and (1.56).

Remark 1.7.2. *Results (3)-(4) technically show stability in terms of finite pointwise data sets $\{\omega_j\}_{j=1}^N$, $\{z_j\}_{j=1}^N$ in l^1 norm, however, by the equivalence of norms in finite dimensions, the same results, but with different bounds, also hold for l^p norms, for any $p \in \mathbb{N}_+$ and $p = \infty$.*

1.8 Numerical illustrations and algorithmic aspects

To illustrate the results of Sections 1.4-1.5 and estimate practical computational parameters, we perform the following numerical simulations. First of all, without loss of generality, choose $J = \left\{e^{i\theta} : \theta \in [-\theta_0, \theta_0]\right\}$ for some fixed $\theta_0 \in (0, \pi)$. In order to invert the Toeplitz operator in (1.23) in a computationally efficient way, we realize projection of equation (1.27) onto finite dimensional (truncated) Fourier basis $\left\{z^{k-1}\right\}_{k=1}^Q$ for large enough $Q \in \mathbb{N}_+$ and look for approximate solution in the form

$$g(z) = \sum_{k=1}^{Q} g_k z^{k-1}. \tag{1.105}$$

Introducing, for $m, k \in \{1, \ldots, Q\}$,

$$A_{k,m} := \begin{cases} \dfrac{\sin(m-k)\theta_0}{\pi(m-k)}, & m \neq k, \\ \theta_0/\pi, & m = k, \end{cases} \qquad A := \left[A_{k,m}\right]_{k,m=1}^Q, \tag{1.106}$$

$$s_k := \left\langle \left(\bar{b}\left(f - \psi\right) \vee (1+\mu)\,\bar{b}\left(h - \psi\right)\right), e^{i(k-1)\theta} \right\rangle_{L^2(0,2\pi)}, \qquad \boldsymbol{s} := \left[s_k\right]_{k=1}^Q, \tag{1.107}$$

the projection equation

$$\left\langle (1+\mu\phi)\, g - P_+\left(\bar{b}\,(f-\psi) \vee (1+\mu)\,\bar{b}\,(h-\psi)\right), z^{k-1}\right\rangle_{L^2(\mathbb{T})} = 0$$

becomes the vector equation (if we employ 1 to denote the identity $Q \times Q$ matrix)

$$(1+\mu A)\,\boldsymbol{g} = \boldsymbol{s}, \qquad \boldsymbol{g} := [g_k]_{k=1}^{Q} \tag{1.108}$$

with a real symmetric Toeplitz matrix which is computationally cheap to invert: depending on the algorithm, asymptotic complexity of inversion may be as low as $\mathcal{O}\left(Q \log^2 Q\right)$ (see [15] and references therein).

Now, in order to numerically demonstrate the monotonicity results (1.56) for e and M_0 with respect to the parameter μ and to compare the behavior with that of series expansions (1.71)-(1.72), we run simulation for the following set of data. We choose $N = 5$, $\theta_0 = \pi/3$, and

$$f(\theta) = f_0(\theta) + \frac{\epsilon}{\exp(i\theta) - 0.4 - 0.3i}, \quad f_0(\theta) := \exp(5i\theta) + \exp(2i\theta) + 1 \in \mathcal{A}^{\psi,b}$$

(when the parameter $\epsilon \neq 0$, obviously, $f \in L^2(I)$ does not extend inside the disk as a H^2 function). Further, f_0 is the restriction of the function $z^5 + z^2 + 1$ satisfying pointwise interpolation conditions (1.13) for points $\{z_j\}_{j=1}^5$ and values $\{\omega_j\}_{j=1}^5$ chosen as given in Table 1.1. We also take $h \in L^2(J)$ as

$$h(\theta) = \frac{1}{\exp(i\theta) - 0.5i}.$$

Based on the points $\{z_j\}_{j=1}^5$, we construct the Blaschke product according to (1.9) with the choice of constant $\phi_0 = 0$ (obviously, final physical results should not depend on a choice of this auxiliary parameter which is also clear from the solution formula (1.53)). The interpolant ψ was chosen as (1.46). Series expansions (1.71)-(1.72) are straightforward to evaluate numerically since $F(k)$ involves the quantity ξ_0 given by (1.70). The projections P_+ there are computed by performing non-negative-power expansions as (1.105) whereas ϕ^k is simply iterative multiplication of the first Q Fourier coefficients of ξ_0 by the Toeplitz operator matrix (1.106). Such iterations are extremely cheap to compute once the matrix A is diagonalized.

Figures 1.8.1-1.8.2 illustrate approximation errors on I and discrepancies on J versus the parameter μ for different values of ϵ when the dimension of the solution space is fixed to $Q = 20$. Number of terms in the series expansions (1.71)-(1.72) was kept fixed at $S = 10$ (such that it is the maximal power of μ in the series). It is remarkable that even such a low number of terms gives bounds which are in very reasonable agreement with those computed from solution up to relatively close neighborhood of $\mu = -1$. On Figure 1.8.3, we further investigate change of deviation of the series expansion from the solution computed numerically (which is taken as a reference in this case, see the discussion in the next paragraph) as more terms are taken into account in the expansions.

Figure 1.8.4 shows variation of the results with respect to truncation of the solution basis while the parameter $\epsilon = 0.5$ is kept fixed. Errors are compared to results obtained for $Q = 50$ which is taken as reference. We conclude that a choice of Q between 10 and 20 is already sufficiently good for practical purposes. In particular, we can regard the numerical computation results obtained for $Q = 20$ as those corresponding to faithful solution so to compare them with what follows from the series expansions (1.71)-(1.72). Clearly, a choice of $Q < N = 5$ does not make sense since, according to Lemma 1.4.1, the interpolant ψ can be chosen as a polynomial which, under such a restriction, will not even be able to meet all pointwise constraints.

Finally, on Figure 1.8.5, we plot auxiliary quantities $F(k)$ and $kF(k)$ versus k which fundamentally enter the series expansions (1.71)-(1.72). In such a computation of multiple iterative action of the Toeplitz operator ϕ on a fixed H^2 function mentioned above, we used high value of $Q = 50$ to prevent possible accumulation of error steming from the truncation to a finite dimensional basis. The first quantity $F(k)$ demonstates the expected decay to zero, while the second one shows that the decay is not fast enough to produce a summable series (that is, $F(k) \neq o(1/k)$ as $k \to \infty$) which illustrates the sharpness of Lemma 1.5.1 and, on the other hand, is consistent with blow-up of $M_0^2(\mu)$ near $\mu = -1$.

Suggested computational algorithm Even though Figure 1.8.3 shows good accuracy of approximation $e(\mu)$ and $M_0^2(\mu)$ from the series expansions (1.71)-(1.72), it is clear, by nature of such expansions, that the convergence slows down as μ gets closer to -1, and hence, for the genuine values, the number of terms in the series should be increased dramatically. However, as it was mentioned, the quantities $F(k)$ are very cheap to compute. It remains only to estimate S, that is the number of terms in series for the accurate approximation of $e(\mu)$ and $M_0^2(\mu)$, but it suffices to perform such a calibration only once, namely, for the lowest value of μ in the computational range. This suggests the following computational strategy:

1. Decide on the lowest value of the Lagrange parameter μ_0 by checking the approximation rate computed from solving the system (1.108). The quantity $e(\mu_0)$ will then be the best approximation rate on I.

2. Determine the number of terms S by comparing the approximation rate with that evaluated from the expansion (1.72) for μ_0.

3. Fix S, precompute the values $F(k)$, $k = 1, \ldots, S$. Vary the parameter μ and evaluate the approximation and blow-up rates from the expansions (1.71)-(1.72) in order to find a suitable trade-off.

z	ω
$0.5+0.4i$	$0.9852+0.3752i$
$-0.3+0.3i$	$1.0097-0.1897i$
$0.2+0.6i$	$0.7811+0.2362i$
$0.2-0.5i$	$0.8328-0.1852i$
$0.8-0.1i$	$1.9069-0.3584i$

Table 1.1: Interior pointwise data

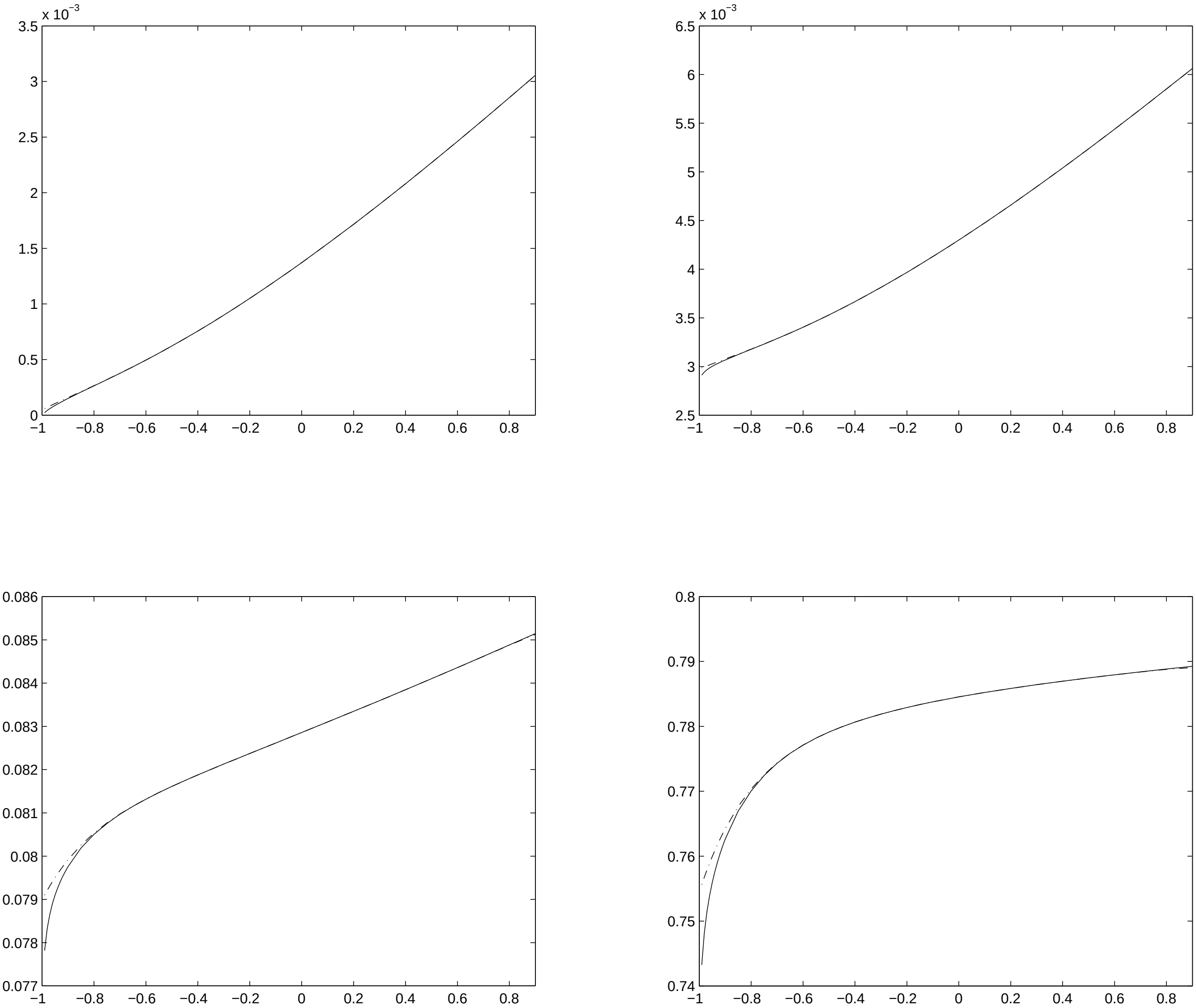

Figure 1.8.1: Relative approximation error on I: $e(\mu) / \|f\|_{L^2(I)}$ from solution (solid) and series expansion (dash-dot) for ϵ=0 (top left), $\epsilon = 0.1$ (top right), $\epsilon = 0.5$ (bottom left), $\epsilon = 2$ (bottom right).

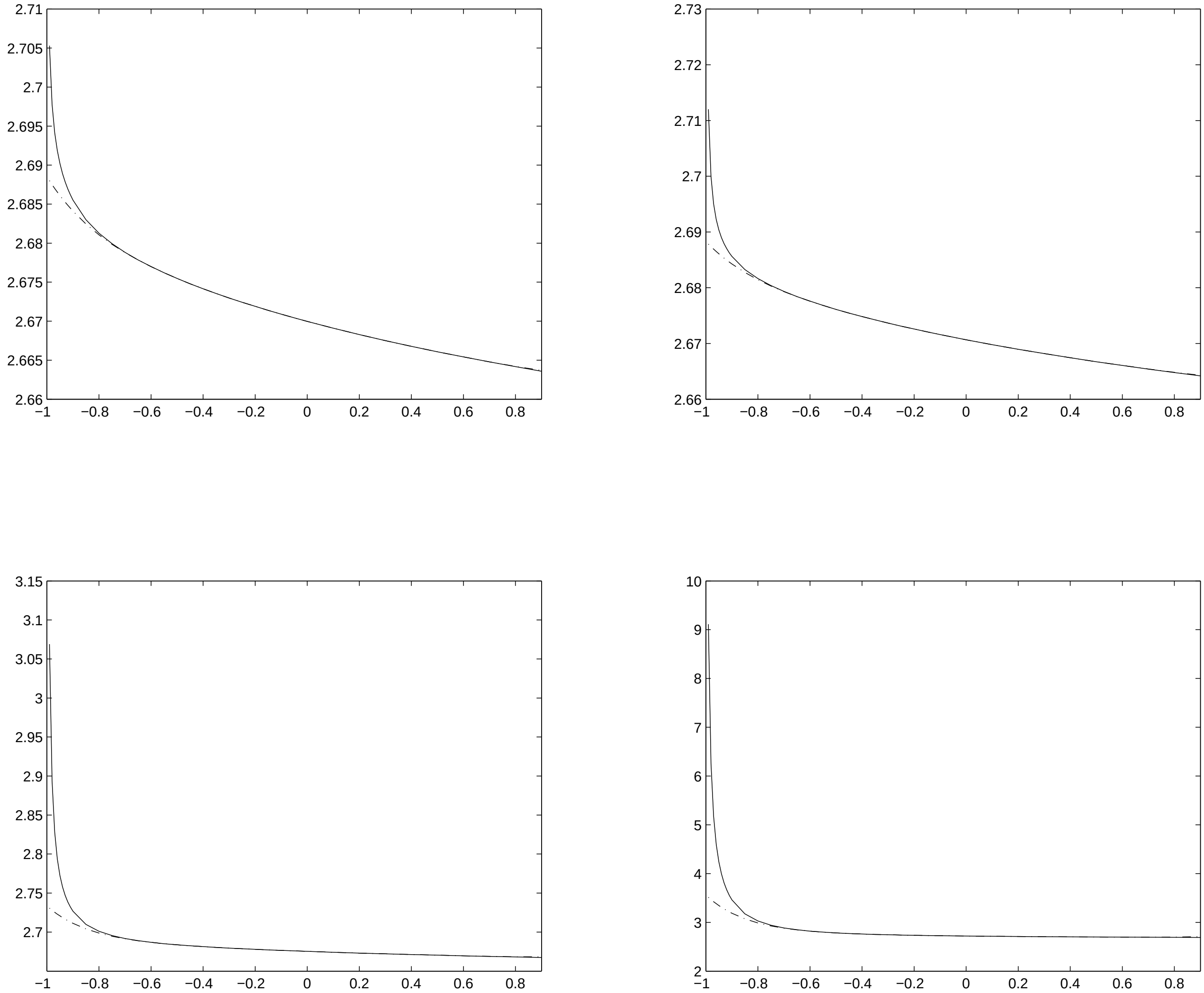

Figure 1.8.2: Relative discrepancy on J: $M_0^2(\mu) / \|h\|_{L^2(J)}$ from solution (solid) and series expansion (dash-dot) for ϵ=0 (top left), $\epsilon = 0.1$ (top right), $\epsilon = 0.5$ (bottom left), $\epsilon = 2$ (bottom right).

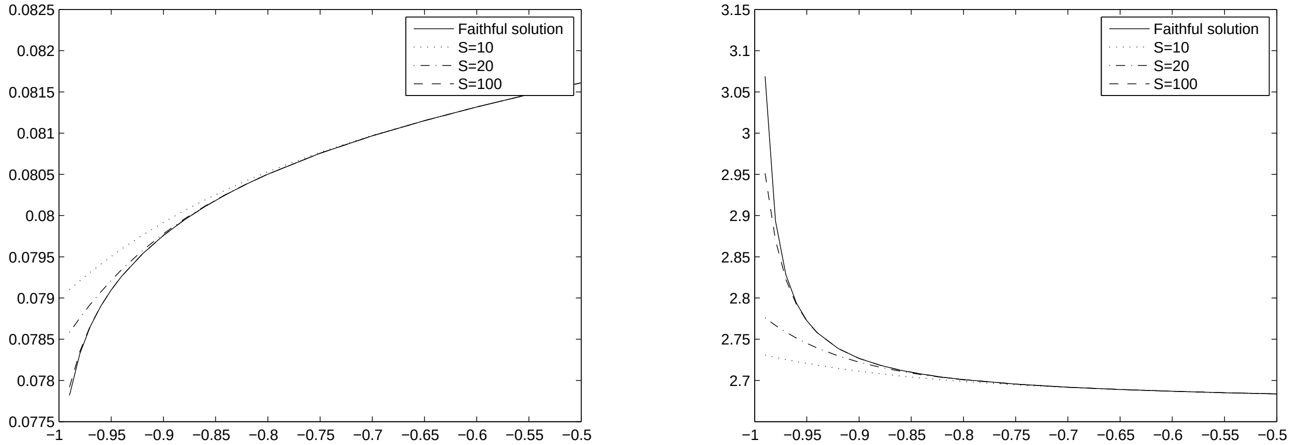

Figure 1.8.3: Relative approximation error on I (left) and relative discrepancy error on J (right).

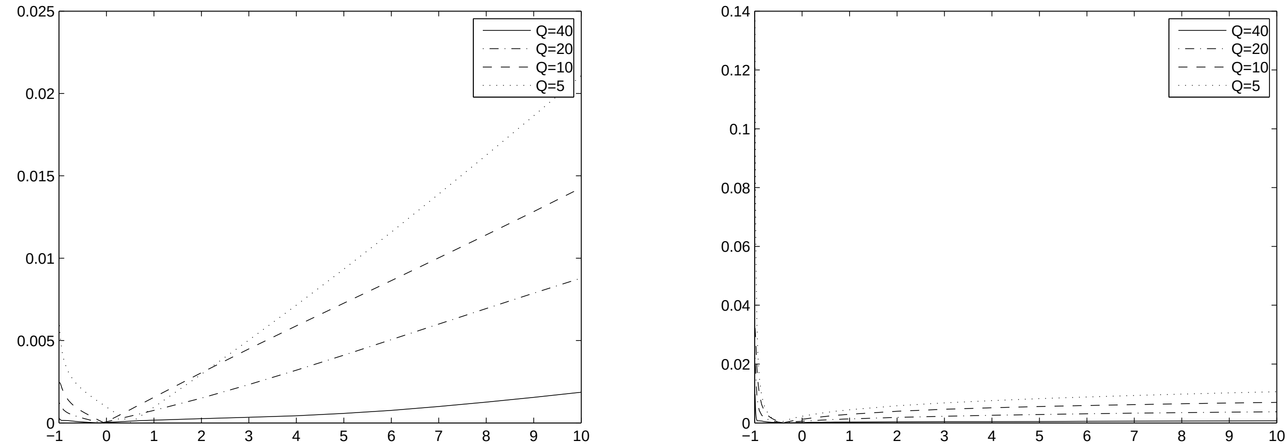

Figure 1.8.4: Errors on I (left) and J (right) compared to results for $Q = 50$.

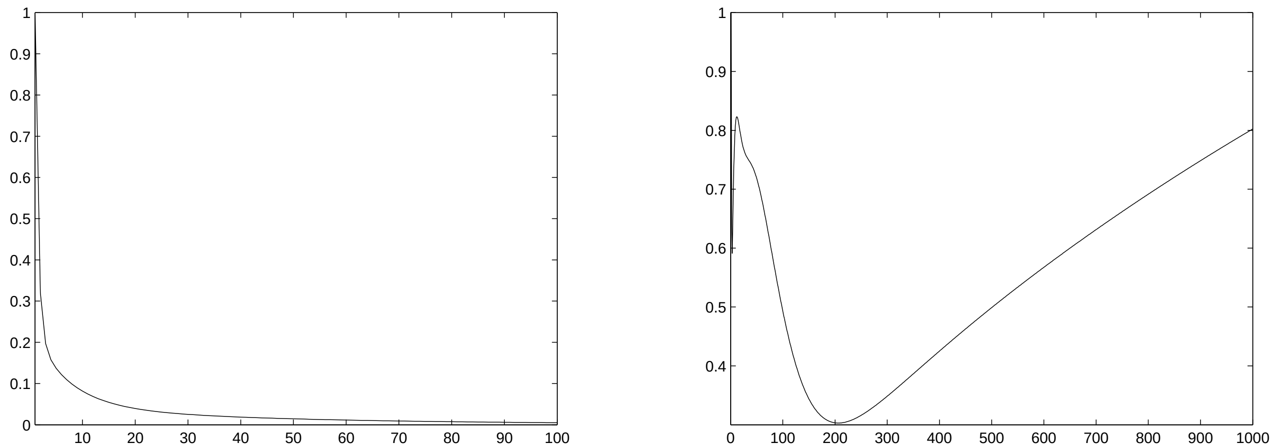

Figure 1.8.5: Auxiliary quantities $F(k)$ and $kF(k)$ computed with $Q = 50$.

APPENDIX

Theorem. *(Hartman-Wintner)*

Let $\xi \in L^{\infty}(\mathbb{T})$: $\mathbb{T} \to \mathbb{R}$ be a symbol defining the Toeplitz operator $T_\xi : H^2 \to H^2 : F \mapsto T_\xi(F) = P_+(\xi F)$. Then, the operator spectrum is $\sigma(T_\xi) = [\text{ess inf}\, \xi,\ \text{ess sup}\, \xi] \subset \mathbb{R}$.

Proof. We give a proof combining ideas from both [17, Thm 7.20] and [35, Thm 4.2.7] in a way such that it is short and self-consistent.

First of all, since ξ is a real-valued function, T_ξ is self-adjoint, and hence $\sigma(T_\xi) \subset \mathbb{R}$.
Now, to prove the result, we employ definition of $\sigma(T_\xi)$ as complement of resolvent set, namely, given $\mu \in \mathbb{R}$, we aim to show that the existence and boundedness of $(T_\xi - \mu I)^{-1}$ on H^2 (i.e. when μ is in the resolvent set) necessarily imply that either $\xi - \mu > 0$ or $\xi - \mu < 0$ a.e., in other words, $(\xi - \mu)$ must be strictly uniform in sign a.e. on $\mathbb{D}$.
Assume μ is fixed so that the inverse of $(T_\xi - \mu I)$ exists and bounded on the whole H^2, in particular, on constant functions. This means that there is $f \in H^2$ such that

$$T_{\xi-\mu} f = (T_\xi - \mu I) f = 1.$$

For any $n \in \mathbb{N}_+$, denoting f_k the coefficients of Fourier expansion of f on $\mathbb{T}$, let us evaluate

$$\langle T_{\xi-\mu} f, z^n f \rangle_{L^2(\mathbb{T})} = \langle 1, z^n f \rangle_{L^2(\mathbb{T})} = \langle z^n, \bar{f} \rangle_{L^2(\mathbb{T})} = \sum_{k=0}^{\infty} f_k \int_0^{2\pi} e^{i(n+k)\theta} d\theta = 0.$$

On the other hand, since $z^n f \in H^2$, we have

$$\langle T_{\xi-\mu} f, z^n f \rangle_{L^2(\mathbb{T})} = \langle (\xi - \mu) f, z^n f \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} (\xi - \mu) |f|^2 \bar{z}^n d\sigma,$$

and thus

$$\int_{\mathbb{T}} (\xi - \mu) |f|^2 z^{-n} d\sigma = 0, \quad n \in \mathbb{N}_+,$$

which implies that $(\xi - \mu)|f|^2$ cannot be an analytic function on $\mathbb{D}$ unless it is constant.
However, since ξ and μ are real-valued, taking conjugation yields

$$\int_{\mathbb{T}} (\xi - \mu) |f|^2 z^{n} d\sigma = 0, \quad n \in \mathbb{N}_+,$$

which prohibits $(\xi - \mu)|f|^2$ being non-analytic on $\mathbb{D}$ either. Therefore, $(\xi - \mu)|f|^2 = \text{const}$, and hence $(\xi - \mu)$ has constant sign a.e. on $\mathbb{D}$ that proves the result. $\square$

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PART 2

On a spectral problem for the truncated Poisson operator

2.1 Introduction

2.1.1 The problem formulation and motivation

For h, $a > 0$, let us consider the following homogeneous Fredholm integral equation of the second kind [57]

$$\frac{h}{\pi}\int_{-a}^{a}\frac{f(t)}{(x-t)^2+h^2}dt = \lambda f(x), \quad x \in (-a,a), \tag{2.1}$$

that is, a problem of finding eigenfunctions of the integral operator $P_h\chi_A$: $L^2(A) \to L^2(A)$, where

$$P_h[f](x) := (p_h \star f)(x) = \frac{h}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(x-t)^2+h^2}dt, \tag{2.2}$$

$$p_h(x) := \frac{h}{\pi}\frac{1}{x^2+h^2}, \tag{2.3}$$

and χ_A is the characteristic function of the interval $A := (-a,a)$.

Our motivation for this study lies in the practical possibility of efficient interpolation and extrapolation of data available on A from pointwise measurements, in a limited area, of harmonic fields obeying integral equations with a Poisson type of kernels[1]. As it will be shown, solutions (eigenfunctions) to (2.1) consitute a complete orthonormal set in $L^2(A)$ and hence can be used to expand the partially available data. Such an expansion is expected to be rapidly convergent since ideal data satisfy integral equations with a similar kernel. Moreover, since each eigenfunction is continous and naturally extends to the whole $\mathbb{R}$, so are their finite sums. Therefore, the expansion over such a basis furnishes natural continuation of the data while basis elements are adapted to the

[1]Strictly speaking, the real physical problem in the context of inverse magnetization [3], which we are concerned with, involves three-dimensional kernels, while we consider here its two-dimensional analog.

geometry and physical structure of the problem.

Eigenfunctions of the $P_h\chi_A$ operator naturally arise in Karhunen-Loève expansion of a stationary stochastic process with covariance function given by (2.3) (see, for instance, [9, Ch. 21]).

A non-homogeneous version of the equation (2.1) is encountered in numerous physical problems. In [42], Love reduced an old problem of determining capacitance of a circular condenser to an integral equation that now bears his name, and established existence and uniqueness of its solution. Love's equation is, in fact, (2.1) with $\lambda = \pm 1$ and presence of a non-homogeneous term that is constant. The positive sign corresponds to the situation of oppositely charged condenser plates while the negative one to that of discs of the same charge. A closely related problem is that of motion of viscous fluid between two coaxial slowly rotating discs. The underlying integral equation differs only in the form of the right-hand term (it is now linear rather than constant) [8]. Different signs here correspond to the direction of rotation (the same or opposite angular velocities). In both problems, h is essentially the distance between the disks. The integral equation can be solved numerically by iteration (the integral operator is contractive); however, when the separation parameter h is small, the convergence of Neumann series slows down. In this case, the problem has been studied asymptotically [24, 25]. The ultimate goal of those works was the computation of certain scalar quantities - capacitance (average of the solution of electrostatic Love's equation) or torque on the discs (first algebraic moment of the solution of hydrodynamic Love's equation). Later on, few-term asymptotic expansions were obtained without reduction to Love's equation [38, 39, 62] and, over the years, capacitance computations have been advanced to higher orders [7].

In parallel to that, it was discovered that an analog of Love equation, i.e. (2.1) with a constant term, appears in one-dimensional models of quantum gases with point interaction. In this context, it is known either as Lieb, Lieb-Liniger or Gaudin equation. The cases $\lambda = 1$ (Lieb-Liniger equation) and $\lambda = -1$ (Gaudin equation) correspond to chains of repulsive Bose and attractive Fermi gases, respectively. The equation is formulated for the density of quasi-momenta, and of particular interest are the zeroth- and second-order algebraic moments of the solution determining density of particles and average energy, respectively [18, 40]. In a similar framework, the same equation with $\lambda = -1$ and Poisson kernel itself as inhomogeneous term occurs in modelling of antiferromagnetic insulators [20]. Scalar quantities of interest here are solution average and solution integral against Poisson kernel. Endeavours to obtain higher order asymptotic expansion for scalar quantities arising in such models continue through the years still remaining a contemporary topic of research [26, 28, 65, 66, 72].

In the present work, we consider (2.1) which is a more general analog of all mentioned equations in a sense that it is neither restricted to $\lambda = \pm 1$ nor to a particular form of the non-homogeneous term. In principle, once all solutions (eigenfunctions) of the homogeneous equation are found, a resolvent kernel can be constructed which allows solving any instance of the Love equation with arbitrary non-homogeneous term and any value of λ in $\mathbb{R}\setminus(0,1)$ and almost any λ inside the interval $(0,1)$. However, often the outline of a solution procedure can be repeated for a non-homogeneous version of the equation yielding a more direct form of the solution.

While studying equation (2.1), we discover amusing properties of its solutions (see Section 2.1.1) and derive asymptotic approximations in different range of parameters.

It is worth noting that apart from numerous applications mentioned above, (2.1) is an interesting integral equation problem on its own. Even though, from numerical point of view, this equation is elementary to solve as it is on a finite interval and has a continuous kernel[2], (2.1) is a rare example of an integral equation with an extremely simple kernel whose solutions almost nothing is known about analytically. For example, to the best of our knowledge, even basic multiplicity properties have not been studied for the $P_h\chi_A$ operator. The reason for this neglect in the literature despite the interest in physical applications and harmonic analysis is perhaps the fact that the equation (2.1) wickedly evades well-known techniques [57], including even those for constructing asymptotic solutions, and hence very little conclusive results could be obtained relying on standard methods.

The work [36] considers a class of convolution integral equations on finite intervals with even kernels which are essentially Laplace transforms of functions supported on a subset of the positive half-line. By introducing a non-homogeneous term with an additional parameter, the equation was transformed into a system of singular integral equations for the so-called Chandrasekhar's X- and Y-functions, from which a form of the solution has been deduced due to analyticity constraints. Remarkably, it turned out that odd eigenfunctions were essentially sines, and even ones were cosines, but the frequencies and additional constant term have been found only in terms of solutions of auxiliary integral equations whose closed-form solution was not known. The analysis has been performed for the simplified case when the inverse Laplace transform of a kernel is a positive function with remark that such an assumption is not crucial. However, each zero of the auxiliary function has to be treated separately and, even though it is possible, computations quickly become much more cumbersome [37]. In the present case, the auxiliary function related to the Poisson kernel has infinitely many zeros which hinder further analysis despite the fact that equation (2.1) formally falls into the considered class. Interestingly enough, numerical computations demonstrate that the set of eigenfunctions possesses a Sturm-Liouville property (k-th eigenfunction has $k-1$ zeros), essentially resembling sines and cosines but, in fact, are neither of them due to a boundary-layer correction term which becomes larger near the interval endpoints for eigenfunctions of higher indices. This entails that the limitation of the method of [36] resulting in pure sine and cosine eigenfunctions is not only technical.

Oscillatory properties of eigenfunctions of integral operators are, in general, not uncommon. The class of such integral kernels is known as oscillation kernels [57, Sect. 13.7-4]. These kernels are essentially Green functions of some ODE boundary values problems. Finding an underlying differential equation problem is a lucrative way of attacking an integral equation. However, for convolution equations, the class of kernels producing an integral operator that commutes with some differential one (and hence reduces an integral equation to solving an ODE problem) is rather small and can be roughly described as quotient of two sine/sinc/sinh/sinch kernels with different scaling [21, 43, 76]. However, it is known, that some integral equations possess the Sturm-Liouville structure merely asymptotically, meaning that spectral properties for only high-order eigensolutions behave as those corresponding to a differential operator [53, 70].

The oscillatory property of eigenfunctions is also predicted in [31] by applying matched asymptotic expansion technique to integral integral equations on a unit interval with a small parameter (equivalently, large interval size) in the kernel corresponding to the ratio h/a being small in the present case. Even though this important work lacks rigorous analysis, it is remarkable in that it establishes that integral equations with a wide range of kernels can

[2]Therefore, Love's equation is often a benchmark for testing new numerical methods.

be, for non-small eigenvalues, approximately reduced to a constant coefficient second order differential equation. Nevertheless, in case of (2.1), such an approximation is inapplicable since the Poisson kernel fails to have finite second algebraic moment on the whole real axis. Moreover, the authors consider exactly (2.1) transforming it to an approximate singular integral equation[3] which "appears too difficult to solve explicitly".

A different type of analysis, but again with connection to a Sturm-Liouville problem, has been performed in [23] under more stringent assumptions on the kernel: apart from finiteness of second derivative at the origin of its Fourier transform (equivalent to the assumption of non-vanishing second moment of the kernel), the kernel was required to have an exponential decay at infinity, which is typical for application of the classical Wiener-Hopf technique to solve integral equations [47]. Therefore, the general difficulty in case of (2.1) can be described as lack of decay of the kernel and severe non-analyticity of its Fourier transform (symbol): rational or Gaussian type of symbols would be much more tractable [14, 15, 34, 55].

We should also mention that a number of asymptotic results (again when h is small) have been obtained for non-homogeneous equations by a method specific to the case $\lambda = 1$. This relies on the fact that the Poisson kernel forms an approximate identity and so, viewing the operator P_h as a perturbation of the identity, its expansion cancels out the non-integral term $f(x)$ in (2.1) yielding a Fredholm equation of the first kind. The inverse of this new integral operator can be constructed giving an approximate solution for this version of the equation [27, 69, 75].

A relevant work is that of Pollard [56], where he studied inversion of Poisson transforms of measures on the whole real line. The obtained results are valid beyond absolutely continuous measures, namely, Pollard requires a measure to be only of bounded variation on each finite interval which, in particular, makes his result applicable to truncated Poisson integrals. The exact inversion is constructed as a limiting process applied to an integro-differential operator of infinite order. Even though this provides a formal solution to the Fredholm integral equation of the first kind, it does not seem to help in solving an eigenvalue problem as the inverse Poisson transform is too complicated to give an insight into a problem arising after application of this transform to both sides of (2.1).

Another common approach for obtaining solutions of integral equations in a closed form is their reduction to a Riemann-Hilbert problem of conjugation for analytic functions [1]. The difficulties related to these reformulations associated with (2.1) will be mentioned at the end of Section 2.2.

The present work is mainly aimed at analytical construction of asymptotic representations of eigenfunctions when the geometrical parameter h/a is either large or small. When $h/a \gg 1$, our idea is to approximate the kernel of integral equation by the one that admits a commuting differential operator, and hence reduce the issue to solving a boundary-value problem for ODE. The latter can be solved in terms of special functions: its solutions are essentially spheroidal wave functions [51, 63]. The case $h/a \ll 1$ is more difficult and interesting. It requires rather refined analysis to reduce a problem to a certain integro-differential equation on a half-line. This equation can be solved by approximating it with a Wiener-Hopf type of integral equation for which the closed-form solution can be constructed using reduction to a Riemann-Hilbert problem. The solution for the half-line problem can then be explicitly continued back to the interval tracing the bounds for the approximation error. As far as the the case

[3] At the end of Section 2.3, we rigorously obtain this hypersingular equation by other means.

$h/a \gg 1$ is concerned, even though the range of validity of the approximation was not estimated, we expect it to be good for large eigenfunctions as well since, according to [74], it can be said *a priori* that both the original (2.1) and the approximate integral equation, which leads to an ODE, possess infinitely many eigenvalues having essentially the same asymptotics for their large indices.

For $h/a \ll 1$, we additionally outline other possible computational strategies. Of particular interest is the one based on the direct operator approximation. This leads to a hypersingular integral equation well-known in fracture mechanics theory and air-flow modelling as Prandtl lifting line equation whose analytical solution has been a long-standing problem [10]. We also point out another connection with the integral equation known as Keldysh-Lavrentiev equation which arises in underwater wing motion [54].

Finally, we plot the obtained asymptotical solutions for the both cases $h/a \ll 1$ and $h/a \gg 1$ and compare them with results of numerical solution of the integral equation.

2.1.2 Main properties

It can be easily checked that since the kernel $p_h(x)$ is an even and real-valued function, the operator $P_h\chi_A$ is self-adjoint, and because of the regularity $p_h(x-t) \in L^2(A \times A)$, the operator is also compact (as a Hilbert-Schmidt operator), and hence we have the standard spectral result [44]

Theorem 2.1.1. *There exists $(\lambda_n)_{n=1}^{\infty} \in \mathbb{R}$, $\lambda_n \to 0$ as $n \to \infty$ and $(f_n)_{n=1}^{\infty}$ is a complete set in $L^2(A)$.*

Now we refine the spectral properties. In order to do this, we denote the Fourier transform as

$$\hat{f}_0(k) = \mathcal{F}[\chi_A f](k) = \int_{-a}^{a} f(x)\, e^{2\pi i k x} dx, \tag{2.4}$$

and prove the following

Lemma 2.1.1. *Any solution of (2.1) satisfies the energy identity*

$$\int_0^{\infty} k e^{-2\pi h k} \left[\left|\hat{f}_0(k)\right|^2 + \left|\hat{f}_0(-k)\right|^2 \right] dk = \frac{\lambda a}{2\pi h} \left[|f(a)|^2 + |f(-a)|^2 \right]. \tag{2.5}$$

Proof. Let us differentiate (2.1), multiply by characteristic function χ_A and take the Fourier transform to arrive at

$$-2\pi i \int_{\mathbb{R}} \mathcal{K}_a\left(k, \tilde{k}\right) e^{-2\pi h|\tilde{k}|} \tilde{k} \hat{f}_0\left(\tilde{k}\right) d\tilde{k} = \lambda \int_A f'(x)\, e^{2\pi i k x} dx, \tag{2.6}$$

where $\mathcal{K}_a\left(k, \tilde{k}\right) := \hat{\chi}_A\left(k - \tilde{k}\right) = \frac{\sin\left(2\pi a\left(k-\tilde{k}\right)\right)}{\pi\left(k-\tilde{k}\right)}$ is a reproducing kernel for the Paley-Wiener space [60, Ch. 19]

$$PW^a := \left\{ g \in H(\mathbb{C}) \cap L^2(\mathbb{R}) : \quad |g(k)| \le C e^{2\pi a |\mathrm{Im} k|} \text{ for some } C > 0 \right\}, \tag{2.7}$$

that is, for any $g \in PW^a$, we have

$$\int_{\mathbb{R}} \mathcal{K}_a(x,t)\, g(t)\, dt = g(x),$$

and, certainly, $\mathcal{K}_a(x,\cdot)$, $\mathcal{K}_a(\cdot,t) \in PW^a$ for x, $t \in \mathbb{R}$.

Since $\overline{\hat{f}_0'(k)} \in PW^a$, integration of both sides of (2.6) against it yields

$$-2\pi i \int_{\mathbb{R}} e^{-2\pi h|k|} k \overline{\hat{f}_0'(k)} \hat{f}_0(k)\, dk = \lambda \int_A f'(x) \int_{\mathbb{R}} \overline{\hat{f}_0'(k)} e^{2\pi i k x} dk dx,$$

and hence

$$\int_0^\infty e^{-2\pi h k} k \left[\overline{\hat{f}_0'(k)} \hat{f}_0(k) - \overline{\hat{f}_0'(-k)} \hat{f}_0(-k) \right] dk = \lambda \int_A x f'(x) \overline{f(x)} dx.$$

Adding the complex conjugate equation, we obtain

$$\int_0^\infty e^{-2\pi h k} k \frac{d}{dk} \left[\left| \hat{f}_0(k) \right|^2 + \left| \hat{f}_0(-k) \right|^2 \right] dk = \lambda \int_A x \frac{d}{dx} |f(x)|^2 dx. \tag{2.8}$$

Now note that integration of (2.1) against $\overline{f(x)}$, by Parseval's identity, implies

$$\int_{\mathbb{R}} e^{-2\pi h|k|} \left| \hat{f}_0(k) \right|^2 dk = \lambda \int_A |f(x)|^2 dx \quad \Rightarrow \quad \int_A |f(x)|^2 dx = \frac{1}{\lambda} \int_0^\infty e^{-2\pi h k} \left[\left| \hat{f}_0(k) \right|^2 + \left| \hat{f}_0(-k) \right|^2 \right] dk.$$

Performing integration by parts in both sides of (2.8) and employing the last identity, we arrive at the result (2.5). □

Proposition 2.1.1. *For λ, f satisfying (2.1), the following statements hold true*

(a) $\lambda \in (0,1)$, *(b)* $f \in C^\infty\left(\bar{A}\right)$,

(c) All $(\lambda_n)_{n=1}^\infty$ *are simple,* *(d) All* f *are either even or odd, and real-valued (up to a multiplicative constant).*

Proof. *(a)* First, we deduce that $\lambda < 1$ from

$$\lambda \|f\|_{L^\infty(A)} = \sup_{x \in A} \left| \frac{h}{\pi} \int_A \frac{f(t)}{(x-t)^2 + h^2} dt \right| < \|f\|_{L^\infty(A)} \|p_h\|_{L^1(\mathbb{R})} = \|f\|_{L^\infty(A)}. \tag{2.9}$$

To get the lower-bound (and hence show that $P_h \chi_A$ is a positive operator), we apply Parseval's identity, convolution theorem for Fourier transform and positivity of the operator symbol $\hat{p}_h$

$$\lambda \|f\|_{L^2(A)}^2 = \langle p_h \star \chi_A f, \chi_A f \rangle_{L^2(\mathbb{R})} = \left\langle e^{-2\pi h|\cdot|} \widehat{\chi_A f}, \widehat{\chi_A f} \right\rangle_{L^2(\mathbb{R})} > 0.$$

(b) follows by regularity of the kernel which admits differentiation under the integral sign in (2.1) arbitrary many times.

To prove *(c)*, we note that multiplicity of compact operator can only be a finite number [44], and, because of convolution structure, eigensubspaces corresponding to the same eigenvalue must necessarily be closed with respect to the integration operator $A^\star[f] := -i \int_x^a f(t)\, dt$ (see [61, Thm 2.2.3]). Differentiating sufficiently many times we conclude that eigenfunctions must satisfy a homogeneous ODE with constant coefficients and hence be a combination of exponentials which, on the other hand, is not possible since integration against Poisson kernel in the left-hand side of (2.1) gives rise to exponential integrals which cannot cancel each other out while there are no integral terms on the right at all. More formally, suppose that $\sum_{k=0}^N c_k f^{(k)}(x) = 0$ for some $c_k \in \mathbb{C}$, $k = 0, \ldots, N$,

$c_N \neq 0$. Using convolution structure of the kernel implying $\partial_x p_h(x-t) = -\partial_t p_h(x-t)$, we can differentiate both sides of (2.1) and then integrate by parts

$$\lambda f'(x) = -\int_{-a}^{a} f(t)\, \partial_t p_h(x-t)\, dt = -f(a)\, p_h(x-a) + f(-a)\, p_h(x+a) + \int_{-a}^{a} f'(t)\, p_h(x-t)\, dt.$$

Performing this procedure iteratively, we get, for $k \in \{1, \ldots, N\}$,

$$-\sum_{j=0}^{k-1} \left[f^{(k-j-1)}(a)\, p_h^{(j)}(x-a) - f^{(k-j-1)}(-a)\, p_h^{(j)}(x+a) \right] + P_h \left[\chi_A f^{(k)} \right](x) = \lambda f^{(k)}(x).$$

Forming a combination $\sum_{k=0}^{N} c_k f^{(k)}(x)$, it then follows from the ODE that

$$\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} \left[f^{(k-j-1)}(a)\, p_h^{(j)}(x-a) - f^{(k-j-1)}(-a)\, p_h^{(j)}(x+a) \right] = 0.$$

From here, the linear independence of $p_h^{(N-1)}(x-a)$ and $p_h^{(N-1)}(x+a)$ entails that $f(a) = f(-a) = 0$. This last conclusion contradicts the energy identity in Lemma 2.1.1 since then $f(x)$ must be identically zero.

Proof of *(d)* is essentially application of the result *(c)* after observation that the integral operator $P_h \chi_A$ commutes with sign inversion operator and real part evaluation. □

Remark 2.1.1. *The upper bound for the eigenvalues can be improved to*

$$\lambda \leq \frac{2}{\pi} \arctan \frac{a}{h}, \tag{2.10}$$

which directly follows from (2.9) upon replacing $\|p_h\|_{L^1(\mathbb{R})}$ with a sharper estimate

$$\sup_{x \in A} \frac{h}{\pi} \int_A \frac{dt}{(x-t)^2 + h^2} = \frac{1}{\pi} \sup_{x \in A} \left[\arctan \frac{a-x}{h} + \arctan \frac{a+x}{h} \right]$$

and observation that the maximum of the expression is attained at $x = 0$. We note that, in particular, (2.10) implies that $1 - \lambda \gtrsim \frac{2h}{\pi a}$ for $\frac{h}{a} \ll 1$.

It is also known that since the kernel $p_h(x)$ is a restriction of a function analytic in an open set (e.g. an ellipse) around the interval A, the rate of decay of eigenvalues is geometric [41]. This can be quantified further as given by [74]

Proposition 2.1.2. *Denote by $K(x) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2 \theta}}$ the complete elliptic integral of the first kind. Then, as $n \to \infty$, we have*

$$\log \lambda_n \simeq -n\pi \frac{K(\operatorname{sech}(\pi a/h))}{K(\tanh(\pi a/h))}. \tag{2.11}$$

Now we discuss some scaling property and dependence on parameters that we are going to employ further in order to construct approximate solutions.

Let us set $\phi(x) := f(ax)$ for $x \in (-1, 1)$ and $\varphi(x) := f(xh)$ for $x \in (-a/h, a/h)$. Then, by change of variable, we rewrite (2.1) as

$$\frac{\beta}{\pi}\int_{-1}^{1}\frac{\phi(t)}{(x-t)^2+\beta^2}dt=\lambda\phi(x), \quad x\in(-1,1), \tag{2.12}$$

$$\frac{1}{\pi}\int_{-1/\beta}^{1/\beta}\frac{\varphi(t)}{(x-t)^2+1}dt=\lambda\varphi(x), \quad x\in(-1/\beta,1/\beta), \tag{2.13}$$

where $\beta := h/a$, and hence we conclude that ϕ, φ and λ depend only on one parameter - the ratio of h and a.

The latter fact allows us to show monotonicity of eigenvalues with respect to the parameters. This would be the content of Proposition 2.1.3 which, in turn, hinges on Lemma 2.1.2.

Before embarking on proofs of both Lemma 2.1.2 and Proposition 2.1.3, let us make one remark.

Note that, since the spectrum is simple (Proposition 2.1.1), we can order eigenvalues as

$$0<\cdots<\lambda_3<\lambda_2<\lambda_1<1,$$

and denote f_k the eigenfunction corresponding to λ_k, $k\in\mathbb{Z}_+$. In what follows, when no comparison between different eigenvalues/eigenfunctions are made, we will continue writing simply f, λ instead of f_k, λ_k.

We need the following general result.

Theorem 2.1.2. *(Implicit mapping [33, Ch. XIV Thm 2.1])*

Let E, F, G be Banach spaces and consider open subsets $U\subset E$, $V\subset F$ and $\Phi : U\times V\to G$, a map of class C^p, $p>1$. Let $(x_0,y_0)\in U\times V$ and assume that the Fréchet derivative with respect to the second argument $D_2\Phi(x_0,y_0) : \ F\to G$ is a continuous linear map with continuous inverse. Let $\Phi(x_0,y_0)=0$. Then there exists a continuous map $U_0\to V$ defined on an open neighborhood U_0 of x_0 such that $g(x_0)=y_0$ and such that $f(x,g(x))=0$ for all $x\in U_0$. Moreover, if U_0 is taken to be a sufficiently small ball, then g is uniquely determined, and is also of class C^p.

Lemma 2.1.2. *For a, $h>0$, dependence of f and λ on h and a is smooth* [4].

Proof. Fixing $k\in\mathbb{Z}_+$, we consider an eigenfunction f_k corresponding to λ_k which we normalize as $\|f_k\|_{L^2(A)}=1$. By means of application of implicit mapping theorem (Theorem 2.1.2), we are going to prove smoothness of the mapping $h\mapsto(f_k(h),\lambda_k(h))$ in a neighborhood of some fixed $h=h_0>0$. We let $U=\mathbb{R}_+$, $V=L^2(A)\times(0,1)$, $G=L^2(A)\times\mathbb{R}$, $(x_0,y_0)=(h_0,(f_k(h_0),\lambda_k(h_0)))$, and consider the map $\Phi:U\times V\to G$ given by

$$\Phi(h,(f_k(h),\lambda_k(h)))=\left(P_h[\chi_A f_k]-\lambda_k f_k,\ \|f_k\|^2_{L^2(A)}-1\right) \tag{2.14}$$

so that $\Phi(h_0,(f_k(h_0),\lambda_k(h_0)))=0$.

Fréchet derivative of (2.14) with respect to the second argument acting on $(u,\mu)\in L^2(A)\times(0,1)$ is

$$D_2\Phi(h_0,(f_k(h_0),\lambda_k(h_0)))(u,\mu)=\left(P_{h_0}[\chi_A u]-\lambda_k(h_0)u-\mu f_k,\ 2\langle f_k(h_0),u\rangle_{L^2(A)}\right).$$

[4] The idea of this proof is due to L. Baratchart.

We note that $D_2\Phi$ is obviously a bounded (and hence continuous) linear operator as P_{h_0} is for $h_0 > 0$ and, to fulfill the assumptions of Theorem 2.1.2, we need to show that it is also bijective.

Let us show that $D_2\Phi$ is an injection, i.e. that $D_2\Phi\left(h_0, \left(f_k\left(h_0\right), \lambda_k\left(h_0\right)\right)\right)(u, \mu) = (0,0)$ implies vanishing of u and μ.

As follows from Theorem 2.1.1, $\{f_n\left(h_0\right)\}_{n=1}^{\infty}$ is a complete orthonormal set in $L^2(A)$, so we can expand $u = \sum_{n=1}^{\infty} c_n f_n\left(h_0\right) \in L^2(A)$. Since $P_{h_0}\left[\chi_A f_n\left(h_0\right)\right] = \lambda_n\left(h_0\right) f_n\left(h_0\right)$, we have, since $\lambda_n \neq \lambda_k$ for $k \neq n$ (recall simplicity of the spectrum as proven in Proposition 2.1.1),

$$P_{h_0}\left[\chi_A u\right] - \lambda_k\left(h_0\right) u - \mu f_k\left(h_0\right) = 0 \quad \Rightarrow \quad \begin{cases} c_n\left[\lambda_n\left(h_0\right) - \lambda_k\left(h_0\right)\right] = 0, & n \neq k, \\ \mu f_k\left(h_0\right) = 0, \end{cases} \quad \Rightarrow \quad \begin{cases} c_n = 0, & n \neq k, \\ \mu = 0, \end{cases}$$

$$\langle f_k, u\rangle_{L^2(A)} = 0 \quad \Rightarrow \quad c_k = 0,$$

and hence we deduce that $\mu = 0$ and $c_n = 0$ for all $n \in \mathbb{Z}_+$ implying $f_k\left(h_0\right) = 0$.

Establishing surjectivity is tantamount to showing that given $(g, \nu) \in L^2(A) \times \mathbb{R}_+$, we can find u_0, $\mu_0 \in L^2(A) \times (0,1)$ such that $D_2\Phi\left(h_0, \left(f_k\left(h_0\right), \lambda_k\left(h_0\right)\right)\right)\left(u_0, \mu_0\right) = (g, \nu)$. Expanding $g = \sum_{n=1}^{\infty} a_n f_n\left(h_0\right)$, $u_0 = \sum_{n=1}^{\infty} b_n f_n\left(h_0\right)$, we obtain

$$P_{h_0}\left[\chi_A u_0\right] - \lambda_k\left(h_0\right) u_0 - \mu_0 f_k\left(h_0\right) = g \quad \Rightarrow \quad \begin{cases} b_n = \dfrac{a_n}{\lambda_n\left(h_0\right) - \lambda_k\left(h_0\right)}, & n \neq k, \\ \mu_0 = -a_k, \end{cases}$$

$$2\langle f_k, u_0\rangle_{L^2(A)} = \nu \quad \Rightarrow \quad b_k = \nu/2.$$

It remains to verify that formally found coefficients b_n indeed define an L^2 function. By orthonormality of the basis $\{f_n\left(h_0\right)\}_{n=1}^{\infty}$, the last condition is equivalent to the summability $\sum_{n=1}^{\infty} b_n^2 < \infty$ which holds true due to the square summability of $\{a_n\}_{n=1}^{\infty}$ entailed from $g \in L^2(A)$ by Bessel's inequality [60, Thm 4.17], and the fact that $\lambda_n\left(h_0\right) - \lambda_k\left(h_0\right)$ is bounded away from zero (since $\lambda_k > 0$ and according to Theorem 2.1.1 the only accumulation point in the spectrum is zero).

This proves surjectivity and makes Theorem 2.1.2 applicable. As a result, since Φ is an infinitely differentiable map, we get smooth dependence of f_k, λ_k on h. Similarly, choosing the parameter a instead of h, by another application of implicit function theorem, we arrive at the same conclusion concerning the dependence of f_k, λ_k on a. □

Proposition 2.1.3. *For a, $h > 0$, we have $\dfrac{\partial \lambda}{\partial a} > 0$, $\dfrac{\partial \lambda}{\partial h} < 0$, and $\lambda \nearrow 1$ as $h \searrow 0$.*

Proof. Denoting $\alpha := \dfrac{1}{\beta} = \dfrac{a}{h}$, we observe that (2.13) states that φ, which, by Proposition 2.1.1, we can take to be real-valued and of certain parity, belongs to the kernel of the Fredholm operator $\left(P_1\chi_{(-\alpha,\alpha)} - \lambda\right)$ on $L^2(-\alpha, \alpha)$. On the other hand, because of the smooth dependence stated in Lemma 2.1.2, we can take derivatives of both sides of (2.13) with respect to α, and thus obtain

$$\left(P_1\chi_{(-\alpha,\alpha)}-\lambda\right)[\partial_\alpha\varphi](x)=\varphi(x)\,\partial_\alpha\lambda-\frac{1}{\pi}\left(\frac{\varphi(-\alpha)}{(x+\alpha)^2+1}+\frac{\varphi(\alpha)}{(x-\alpha)^2+1}\right)=:\eta(x),\quad x\in(-\alpha,\alpha),$$

which is a statement that $\eta(x)$ belongs to the range of the same operator. By the Fredholm alternative [44] and self-adjointness of the operator, we must have $\eta\in\left(\operatorname{Ker}\left(P_1\chi_{(-\alpha,\alpha)}-\lambda\right)\right)^{\perp}$, i.e. $\int_{-\alpha}^{\alpha}\eta(x)\,\varphi(x)\,dx=0$ giving

$$\|\varphi\|^2_{L^2(-\alpha,\alpha)}\,\partial_\alpha\lambda=\lambda\left[\varphi^2(-\alpha)+\varphi^2(\alpha)\right]=2\lambda\varphi^2(\alpha),$$

a rather general result obtained in [58].

Now, since $\varphi(\alpha)=f(a)$, employing Lemma 2.1.1 and the scaling property $\lambda=\lambda\left(\frac{a}{h}\right)$ discussed above, we conclude that $\partial_a\lambda>0$, $\partial_h\lambda<0$.

To show that $\lambda\nearrow 1$ as $h\searrow 0$, we invoke general Courant-Fischer min-max principle (known also as Rayleigh-Ritz characterization of eigenvalues) [35, Ch. 28 Thm 4] stating that

$$\lambda_k=\max_{S_k}\min_{\substack{u\in S_k,\\ u\neq 0}}\frac{\langle P_h[\chi_A u],u\rangle_{L^2(A)}}{\|u\|^2_{L^2(A)}},\quad k\in\mathbb{Z}_+,\tag{2.15}$$

where S_k is a k-dimensional subspace of $L^2(A)$. By the approximate identity property of the Poisson operator [16, Thm 3.1], we have

$$\langle P_h[\chi_A u],u\rangle_{L^2(A)}\to\|u\|^2_{L^2(A)}\quad\text{as } h\searrow 0$$

establishing the result. □

Let us also note that the limit $\lambda\nearrow 1$ as $h\searrow 0$ is in agreement with the estimate (2.10) suggesting its certain sharpness. Indeed, for the very first eigenvalue λ_1 the bound from below can be easily obtained. This is a consequence of the min-max characterization (2.15) which in this case, by choosing $u=1$, implies

$$\begin{aligned}\lambda_1 &= \max_{\substack{u\in L^2(A),\\ u\neq 0}}\frac{\langle P_h[\chi_A u],u\rangle_{L^2(A)}}{\|u\|^2_{L^2(A)}}\geq\frac{1}{2a}\int_{-a}^{a}\int_{-a}^{a}p_h(x-t)\,dtdx\\ &= \frac{1}{2\pi a}\int_{-a}^{a}\left(\arctan\frac{a-x}{h}+\arctan\frac{a+x}{h}\right)dx=\frac{1}{\pi a}\int_{-a}^{a}\arctan\frac{a+x}{h}dx,\end{aligned}$$

and hence

$$\lambda_1\geq\frac{2}{\pi}\arctan\frac{2a}{h}-\frac{h}{2\pi a}\log\left(1+\frac{4a^2}{h^2}\right).\tag{2.16}$$

The obtained bounds (2.10) and (2.16) sandwich λ_1 in an interval whose size is as small as $\mathcal{O}\left(\frac{h}{a}\log\frac{a}{h}\right)$ for $\frac{h}{a}\ll 1$.

Note that the quantified version of the behavior $\lambda\nearrow 1$ as $h\searrow 0$ can be obtained from (2.11) using the asymptote of the complete elliptic integral $K(1-x)\simeq-\frac{1}{2}\log(2x)$, $0<x\ll 1$, which is derived from [50,

(19.12.1)], and the evident fact $K(0) = \pi/2$. Namely, we deduce that

$$\log \lambda_n \simeq -\frac{\pi h n}{2a}, \quad n \gg 1,\ h \ll 1. \tag{2.17}$$

We conclude by listing few other more subtle solution properties and features of the problem.

Define, for $x \in \mathbb{R} \backslash A$,

$$f(x) = \frac{h}{\lambda \pi} \int_A \frac{f(t)}{(x-t)^2 + h^2} dt \tag{2.18}$$

so that the validity of the equation (2.1) persists on the whole $\mathbb{R}$. Then, integrating it against a function $g(\cdot, t) \in H(\Pi_\pm)$ boundedly analytic in the first variable in the upper Π_+ or the lower Π_- half-plane, with help of residue calculus, we obtain

$$\int_A f(x) g(x \pm ih, t)\, dx = \lambda \int_{-\infty}^{\infty} f(x) g(x, t)\, dx, \tag{2.19}$$

which suggests the existence of an efficient transformation of the problem by means of wisely chosen $g(x,t)$.

In particular, simply taking $g(x,t) \equiv 1$, this gives

$$\int_A f(x)\, dx = \lambda \int_{-\infty}^{\infty} f(x)\, dx. \tag{2.20}$$

Due to positivity of operator $P_h \chi_A$ and variational characterization of eigenvalues due to Rayleigh quotient maximization (recall min-max principle in the proof of Proposition 2.1.3), the eigenfunction corresponding to the largest eigenvalue can be chosen to be positive on A. Then, its extension outside A must be positive as well, and so (2.20) can be rewritten as $\lambda = \frac{\|f\|_{L^1(A)}}{\|f\|_{L^1(\mathbb{R})}}$. This spectral concentration property is slightly reminiscent to that of prolate spheroidal functions [51].

Another interesting property is double orthogonality of Fourier transforms of eigenfunctions of certain parity: let f_l, f_m be both odd or even eigenfunctions, then, by Parseval's identity, the Fourier transforms $\hat{f}_l(k) = \mathcal{F}[\chi_A f_l](k)$, $\hat{f}_m(k) = \mathcal{F}[\chi_A f_m](k)$ are orthogonal on the half-line with both constant and exponential weights

$$\int_0^\infty e^{-2\pi h k} \hat{f}_l(k) \hat{f}_m(k)\, dk = 0 = \int_0^\infty \hat{f}_l(k) \hat{f}_m(k)\, dk, \quad l \neq m. \tag{2.21}$$

Note that if exponential factor was replaced by a characteristic function, one would obtain a property of double orthogonality with respect to the range of integration. This property is also typical to prolate spheroidal harmonics [63] which suggests a connection that will be established in a further section.

2.2 Some reformulations of the problem

2.2.1 Integral equations in Fourier domain

Most natural way of studying convolution equations is by looking at their Fourier domain formulation. Let us multiply (2.1) by χ_A and apply Fourier transform. We obtain an integral equation with regular kernel on the whole line

$$\int_{\mathbb{R}} \frac{\sin\left(2\pi a\left(k-\tilde{k}\right)\right)}{\pi\left(k-\tilde{k}\right)} e^{-2\pi h|\tilde{k}|} \hat{f}_0\left(\tilde{k}\right) d\tilde{k} = \lambda \hat{f}_0(k), \quad k \in \mathbb{R}, \tag{2.22}$$

where $\hat{f}_0$ is as defined in (2.4).

Recalling (2.7), we have that $\hat{f}_0$, $\frac{\sin\left(2\pi a\left(\cdot-\tilde{k}\right)\right)}{\pi\left(\cdot-\tilde{k}\right)} \in PW^a$ for any $\tilde{k} \in \mathbb{R}$, and so, by analyticity, validity of (2.22) extends to the whole complex plane $\mathbb{C}$.

In particular, we are going to derive an integral equation analogous to (2.22) that will involve instances of $\hat{f}_0$ on the imaginary axis $i\mathbb{R}$.

Let us recall the definition of Cauchy principal value integral. Let $F(t)$ be a function defined on an interval (a, c) such that it is singular at some point $b \in (a, c)$ where it does not have to be absolutely integrable. Then, the principal value integral of F is defined as the following limit

$$⨍_a^c F(t)\, dt := \lim_{\epsilon \searrow 0} \left(\int_a^{b-\epsilon} + \int_{b+\epsilon}^c\right) F(t)\, dt.$$

Now we are in position to formulate

Proposition 2.2.1. *Equation (2.22) is equivalent to the following singular integral equation*

$$\frac{1}{\pi} ⨍_{\mathbb{R}} \frac{\sin(2\pi h t)}{t-\tau} e^{-2\pi a (t-\tau)\operatorname{sgnt}} \hat{f}_0(it)\, dt = \left[\cos(2\pi h\tau) - \lambda\right] \hat{f}_0(i\tau), \quad \tau \in \mathbb{R}, \tag{2.23}$$

or, alternatively, for even (upper sign) and odd (lower sign) solutions

$$e^{2\pi a\tau} ⨍_0^\infty \frac{\sin(2\pi h t)\, e^{-2\pi a t}}{t-\tau} \hat{f}_0(it)\, dt \pm e^{-2\pi a\tau} ⨍_0^\infty \frac{\sin(2\pi h t)\, e^{-2\pi a t}}{t+\tau} \hat{f}_0(it)\, dt = \pi\left[\cos(2\pi h\tau) - \lambda\right] \hat{f}_0(i\tau), \quad \tau \in \mathbb{R}. \tag{2.24}$$

Proof. Let us evaluate the left-hand side of (2.22) at $k = i\tau$, rewriting it as

$$\begin{aligned} \int_{\mathbb{R}} \frac{\sin\left(2\pi a\left(i\tau-\tilde{k}\right)\right)}{\pi\left(i\tau-\tilde{k}\right)} e^{-2\pi h|\tilde{k}|} \hat{f}_0\left(\tilde{k}\right) d\tilde{k} \quad = \quad & \frac{e^{-2\pi a\tau}}{2\pi i} \int_{-\infty}^0 \frac{e^{2\pi(h-ia)\tilde{k}}}{i\tau-\tilde{k}} \hat{f}_0\left(\tilde{k}\right) d\tilde{k} - \frac{e^{2\pi a\tau}}{2\pi i} \int_{-\infty}^0 \frac{e^{2\pi(h+ia)\tilde{k}}}{i\tau-\tilde{k}} \hat{f}_0\left(\tilde{k}\right) d\tilde{k} \\ & + \frac{e^{-2\pi a\tau}}{2\pi i} \int_0^\infty \frac{e^{-2\pi(h+ia)\tilde{k}}}{i\tau-\tilde{k}} \hat{f}_0\left(\tilde{k}\right) d\tilde{k} - \frac{e^{2\pi a\tau}}{2\pi i} \int_0^\infty \frac{e^{-2\pi(h-ia)\tilde{k}}}{i\tau-\tilde{k}} \hat{f}_0\left(\tilde{k}\right) d\tilde{k}. \end{aligned}$$

Now using analyticity of $\hat{f}_0$ and the fact that

$$e^{2\pi i a k} \hat{f}_0(k) = \int_0^{2a} e^{2\pi i x k} f(x-a)\, dx \quad \text{and} \quad e^{-2\pi i a k} \hat{f}_0(k) = \int_{-2a}^0 e^{2\pi i x k} f(x+a)\, dx$$

decay in the upper and the lower half-plane, respectively, we employ Cauchy theorem to deform an integration contour in each integral closing it in such a quadrant in which the integrand decays at infinity[5].

[5]Note that the decay of the integrand at infinity is exponential except on the imaginary axis where it is only algebraic, e.g. $e^{-2\pi a k} \hat{f}_0(-ik) = \int_{-2a}^0 e^{2\pi x k} f(x+a)\, dx = \dfrac{f(a) - e^{-4\pi a} f(-a)}{2\pi k} - \dfrac{1}{2\pi k} \int_{-2a}^0 e^{2\pi x k} f'(x+a)\, dx = \mathcal{O}(1/k)$.

Elaborating the first term

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{2\pi(h-ia)\tilde{k}}}{i\tau-\tilde{k}}\hat{f}_0\left(\tilde{k}\right)d\tilde{k} &= -\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{-i\infty}\frac{e^{2\pi(h-ia)\left(\tilde{k}-\epsilon\right)}}{i\tau-\left(\tilde{k}-\epsilon\right)}\hat{f}_0\left(\tilde{k}-\epsilon\right)d\tilde{k} = -\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{-i\infty}\frac{e^{2\pi(h-ia)\tilde{k}}}{i\tau-\tilde{k}+\epsilon}\hat{f}_0\left(\tilde{k}\right)d\tilde{k}\\ &= \lim_{\epsilon\searrow 0}\frac{1}{2\pi}\int_{-\infty}^{0}\frac{e^{2\pi(h-ia)it}}{i\tau-it+\epsilon}\hat{f}_0(it)\,dt = -\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{2\pi(h-ia)it}}{t-\tau+i\epsilon}\hat{f}_0(it)\,dt,\end{aligned}$$

and so, by Plemelj-Sokhotskii formula (in case of $\tau<0$), we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{2\pi(h-ia)\tilde{k}}}{i\tau-\tilde{k}}\hat{f}_0\left(\tilde{k}\right)d\tilde{k} &= \begin{cases}\frac{1}{2}e^{2\pi(ih+a)\tau}\hat{f}_0(i\tau)-\frac{1}{2\pi i}\fint_{-\infty}^{0}\frac{e^{2\pi(ih+a)t}}{t-\tau}\hat{f}_0(it)\,dt, & \tau<0,\\ -\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{2\pi(ih+a)t}}{t-\tau}\hat{f}_0(it)\,dt, & \tau>0,\end{cases}\\ &= \frac{1}{4}(1-\operatorname{sgn}\tau)\,e^{2\pi(ih+a)\tau}\hat{f}_0(i\tau)-\frac{1}{2\pi i}\fint_{-\infty}^{0}\frac{e^{2\pi(ih+a)t}}{t-\tau}\hat{f}_0(it)\,dt.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{2\pi(h+ia)\tilde{k}}}{i\tau-\tilde{k}}\hat{f}_0\left(\tilde{k}\right)d\tilde{k} &= -\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{i\infty}\frac{e^{2\pi(h+ia)\left(\tilde{k}-\epsilon\right)}}{i\tau-\left(\tilde{k}-\epsilon\right)}\hat{f}_0\left(\tilde{k}-\epsilon\right)d\tilde{k} = -\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{i\infty}\frac{e^{2\pi(h+ia)\tilde{k}}}{i\tau-\tilde{k}+\epsilon}\hat{f}_0\left(\tilde{k}\right)d\tilde{k}\\ &= -\lim_{\epsilon\searrow 0}\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{2\pi(h+ia)it}}{i\tau-it+\epsilon}\hat{f}_0(it)\,dt = \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{\infty}\frac{e^{2\pi(h+ia)it}}{t-\tau+i\epsilon}\hat{f}_0(it)\,dt\\ &= -\frac{1}{4}(1+\operatorname{sgn}\tau)\,e^{2\pi(ih-a)\tau}\hat{f}_0(i\tau)+\frac{1}{2\pi i}\fint_{0}^{\infty}\frac{e^{2\pi(ih-a)t}}{t-\tau}\hat{f}_0(it)\,dt,\end{aligned}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{0}^{\infty}\frac{e^{-2\pi(h+ia)\tilde{k}}}{i\tau-\tilde{k}}\hat{f}_0\left(\tilde{k}\right)d\tilde{k} &= \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{-i\infty}\frac{e^{-2\pi(h+ia)\left(\tilde{k}+\epsilon\right)}}{i\tau-\left(\tilde{k}+\epsilon\right)}\hat{f}_0\left(\tilde{k}+\epsilon\right)d\tilde{k} = \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{-i\infty}\frac{e^{-2\pi(h+ia)\tilde{k}}}{i\tau-\tilde{k}-\epsilon}\hat{f}_0\left(\tilde{k}\right)d\tilde{k}\\ &= -\lim_{\epsilon\searrow 0}\frac{1}{2\pi}\int_{-\infty}^{0}\frac{e^{-2\pi(h+ia)it}}{i\tau-it-\epsilon}\hat{f}_0(it)\,dt = \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{-2\pi(h+ia)it}}{t-\tau-i\epsilon}\hat{f}_0(it)\,dt\\ &= \frac{1}{4}(1-\operatorname{sgn}\tau)\,e^{-2\pi(ih-a)\tau}\hat{f}_0(i\tau)+\frac{1}{2\pi i}\fint_{-\infty}^{0}\frac{e^{-2\pi(ih-a)t}}{t-\tau}\hat{f}_0(it)\,dt,\end{aligned}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{0}^{\infty}\frac{e^{-2\pi(h-ia)\tilde{k}}}{i\tau-\tilde{k}}\hat{f}_0\left(\tilde{k}\right)d\tilde{k} &= \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{i\infty}\frac{e^{-2\pi(h-ia)\left(\tilde{k}+\epsilon\right)}}{i\tau-\left(\tilde{k}+\epsilon\right)}\hat{f}_0\left(\tilde{k}+\epsilon\right)d\tilde{k} = \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{i\infty}\frac{e^{-2\pi(h-ia)\tilde{k}}}{i\tau-\tilde{k}-\epsilon}\hat{f}_0\left(\tilde{k}\right)d\tilde{k}\\ &= \lim_{\epsilon\searrow 0}\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{-2\pi(h-ia)it}}{i\tau-it-\epsilon}\hat{f}_0(it)\,dt = -\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{0}^{\infty}\frac{e^{-2\pi(h-ia)it}}{t-\tau-i\epsilon}\hat{f}_0(it)\,dt\\ &= -\frac{1}{4}(1+\operatorname{sgn}\tau)\,e^{-2\pi(ih+a)\tau}\hat{f}_0(i\tau)-\frac{1}{2\pi i}\fint_{0}^{\infty}\frac{e^{-2\pi(ih+a)t}}{t-\tau}\hat{f}_0(it)\,dt.\end{aligned}$$

Collecting the terms, we obtain (2.23). To get to (2.24), it only rests to recall Proposition 2.1.1, that is $f(x)$ must necessarily be either odd or even, and so is $\hat{f}_0(it)$. $\square$

Corollary 2.2.1. *Solution of (2.23) satisfies an infinite number of discrete conditions*

$$\fint_0^\infty \hat{f}_0(it)\sin(2\pi ht)\left[\frac{e^{-2\pi a(t-\tau_m)}}{t-\tau_m} \pm \frac{e^{-2\pi a(t+\tau_m)}}{t+\tau_m}\right]dt = 0, \tag{2.25}$$

where the upper sign in this expression should be chosen for even solutions and the lower sign for odd solutions, and

$$\tau_m := \pm\frac{\arccos\lambda}{2\pi h} + \frac{1}{h}m, \quad m \in \mathbb{Z}, \tag{2.26}$$

where signs are independent of the parity of $\hat{f}_0$.

Proof. This result is an immediate consequence of the obtained equation (2.23). We note that vanishing of the factor in square brackets in the right-hand side on the set of points (2.26) (with arccos denoting the principal value of inverse cosine function so that $\arccos\lambda \in (0, \pi/2)$ for $\lambda \in (0,1)$) imply vanishing of the left-hand side. This is due to the fact that $\hat{f}_0$ is an entire function, and hence, it cannot have poles.

Therefore, we conclude

$$\fint_{\mathbb{R}} \frac{\sin(2\pi ht)}{t-\tau_m} e^{-2\pi a(t-\tau_m)\operatorname{sgn}t}\hat{f}_0(it)\,dt = 0,$$

and invoking parity assumptions, we can reformulate this into (2.25). □

Remark 2.2.1. *The form of the expression (2.25) makes it tempting to factorize the integral operator by invoking convolution theorem for cosine (case of the upper sign) and sine (case of the lower sign) [64]. However, non-distributional versions of such theorems do not apply due to both singularity of the kernel $e^{-2\pi at}/t$ and exponential growth of the function $\hat{f}_0(it)\sin(2\pi ht)$.*

The result of Corollary 2.2.1 will be revisited at the end of Section 2.3 in the context of approximate solution.

2.2.2 Matrix Riemann-Hilbert problems

It is known that solving an integral equation on an interval can be reduced to factorization of matrix of the associated Riemann-Hilbert problem [6], which is typically the only hope for constructing solution for integral equations of the convolution type. We are going to show equivalence of (2.1) to two such problems.

Let us extend equation (2.1) from the interval to the whole line by means of adding an extra term

$$\int_{\mathbb{R}} p_h(x-t)\,f_0(t)\,dt = \lambda f_0(x) + \psi(x), \quad x \in \mathbb{R}, \tag{2.27}$$

where, as before, $f_0(x) = \chi_A(x)\,f(x)$ and

$$\psi(x) := \chi_{\mathbb{R}\setminus A}(x)\int_A f(t)\,p_h(x-t)\,dt.$$

Suppose further that f, and hence f_0, is of certain parity (Proposition 2.1.1). Then, we can write

$$\psi(x) = \psi_0(x-a) \pm \psi_0(-x-a)$$

for some function ψ_0 supported on $\mathbb{R}_+$ and the upper sign corresponding to the even parity of f and the lower one to the odd parity.

Application of Fourier transform to (2.27) now yields

$$\left(e^{-2\pi h|k|}-\lambda\right)\hat{f}_0\left(k\right)=e^{2\pi ika}\hat{\psi}_0\left(k\right)\pm e^{-2\pi ika}\hat{\psi}_0\left(-k\right),\quad k\in\mathbb{R}. \tag{2.28}$$

Denoting H_+ and H_- the spaces of functions holomorphic in the upper and, respectively, the lower half-plane, we note that

$$e^{2\pi ika}\hat{f}_0\left(k\right),\,\hat{\psi}_0\left(k\right)\in H_+,\quad e^{-2\pi ika}\hat{f}_0\left(k\right),\,\hat{\psi}_0\left(-k\right)\in H_-,$$

as follows immediately from the functions support.

Multiplying both sides of equation (2.28) by $e^{-2\pi ika}$, and taking into account the identity $e^{2\pi iak}\hat{f}\left(k\right)=e^{4\pi iak}e^{-2\pi iak}\hat{f}\left(k\right)$, we arrive at the conjugation problem for holomorphic vector functions

$$\mathbf{X}^+\left(k\right)=\begin{pmatrix}\mp e^{-4\pi iak} & e^{-2\pi h|k|}-\lambda\\ 0 & e^{4\pi iak}\end{pmatrix}\mathbf{X}^-\left(k\right), \tag{2.29}$$

where $\mathbf{X}^+\left(k\right):=\left(\hat{\psi}_0\left(k\right),e^{2\pi iak}\hat{f}_0\left(k\right)\right)^T\in H_+$, $\mathbf{X}^-\left(k\right):=\left(\hat{\psi}_0\left(-k\right),e^{-2\pi iak}\hat{f}_0\left(k\right)\right)^T\in H_-$.

Another Riemann-Hilbert formulation is more specific to the structure of the equation (2.1). While (2.29) is a conjugation problem for analytic functions on the real axis, we can also derive a conjugation condition on the imaginary axis.

To this effect, let us observe that $\hat{p}_h\left(k\right)=e^{-2\pi h|k|}$ extends analytically from $\mathbb{R}_+$ to the right half-plane as $\hat{p}_h^r\left(k\right)=e^{-2\pi hk}$, and from from $\mathbb{R}_-$ to the left half-plane as $\hat{p}_h^l\left(k\right)=e^{2\pi hk}$. In the same fashion, we denote the limiting values of $\hat{\psi}_0$ on the imaginary axis from right and left as $\hat{\psi}_0^r$ and $\hat{\psi}_0^l$, respectively. Now, since $\hat{f}$ is entire, (2.28) implies

$$\frac{e^{2\pi iak}\hat{\psi}_0^r\left(k\right)\pm e^{-2\pi iak}\hat{\psi}_0^r\left(-k\right)}{\hat{p}_h^r\left(k\right)-\lambda}=\frac{e^{2\pi iak}\hat{\psi}_0^l\left(k\right)\pm e^{-2\pi iak}\hat{\psi}_0^l\left(-k\right)}{\hat{p}_h^l\left(k\right)-\lambda},\quad k\in i\mathbb{R},$$

and, because of $\hat{p}_h^r\left(-k\right)=\hat{p}_h^l\left(k\right)$, reversing the sign in this equation leads to

$$\frac{e^{-2\pi iak}\hat{\psi}_0^r\left(-k\right)\pm e^{2\pi iak}\hat{\psi}_0^r\left(k\right)}{\hat{p}_h^l\left(k\right)-\lambda}=\frac{e^{-2\pi iak}\hat{\psi}_0^l\left(-k\right)\pm e^{2\pi iak}\hat{\psi}_0^l\left(k\right)}{\hat{p}_h^r\left(k\right)-\lambda},\quad k\in i\mathbb{R}.$$

Moreover, because of $\hat{\psi}_0\left(k\right)\in H_+$, we have $\hat{\psi}_0^r\left(k\right)=\hat{\psi}_0^l\left(k\right)$ for $k\in i\mathbb{R}_+$ and, equivalently, $\hat{\psi}_0^r\left(-k\right)=\hat{\psi}_0^l\left(-k\right)$ for $k\in i\mathbb{R}_-$.

This yields

$$\hat{\psi}_0^l\left(-k\right)=\pm e^{4\pi iak}\left(\frac{\hat{p}_h^l\left(k\right)-\lambda}{\hat{p}_h^r\left(k\right)-\lambda}-1\right)\hat{\psi}_0^l\left(k\right)+\frac{\hat{p}_h^l\left(k\right)-\lambda}{\hat{p}_h^r\left(k\right)-\lambda}\hat{\psi}_0^r\left(-k\right),\quad k\in i\mathbb{R}_+,$$

$$\hat{\psi}_0^r\left(k\right)=\frac{\hat{p}_h^l\left(k\right)-\lambda}{\hat{p}_h^r\left(k\right)-\lambda}\hat{\psi}_0^l\left(k\right)\pm e^{-4\pi iak}\left(\frac{\hat{p}_h^l\left(k\right)-\lambda}{\hat{p}_h^r\left(k\right)-\lambda}-1\right)\hat{\psi}_0^r\left(-k\right),\quad k\in i\mathbb{R}_-,$$

which, being rewritten in the matrix form, furnishes a Riemann-Hilbert problem for the conjugation of two vector functions $\boldsymbol{Y}^r(k) := \left(\hat{\psi}_0^r(k), \hat{\psi}_0^l(-k)\right)^T$ and $\boldsymbol{Y}^l(k) := \left(\hat{\psi}_0^l(k), \hat{\psi}_0^r(-k)\right)^T$ analytic in the left and right half-planes, respectively,

$$\boldsymbol{Y}^r(k) = G(k)\,\boldsymbol{Y}^l(k), \quad k \in i\mathbb{R}, \tag{2.30}$$

with continuous (except at infinity) matrix coefficient

$$G(i\tau) := \begin{cases} \begin{pmatrix} 1 & 0 \\ \pm\dfrac{2ie^{-4\pi a\tau}\sin(2\pi h\tau)}{e^{-2\pi ih\tau}-\lambda} & \dfrac{e^{2\pi ih\tau}-\lambda}{e^{-2\pi ih\tau}-\lambda} \end{pmatrix}, & \tau \in \mathbb{R}_+, \\[2ex] \begin{pmatrix} \dfrac{e^{2\pi ih\tau}-\lambda}{e^{-2\pi ih\tau}-\lambda} & \pm\dfrac{2ie^{4\pi a\tau}\sin(2\pi h\tau)}{e^{-2\pi ih\tau}-\lambda} \\ 0 & 1 \end{pmatrix}, & \tau \in \mathbb{R}_-. \end{cases}$$

Solving a matrix Riemann-Hilbert problem hinges on the possibility of constructing a certain factorization of its matrix coefficient (namely, factorization into the product of two non-singular matrices whose elements can be analytically continued from the boundary line and have at most algebraic growth at infinity). Beyond matrices with rational entries, there are only few classes for which the constructive factorization procedure is available [59]. Presence of exponential factors producing oscillations and discontinuity at infinity already makes inapplicable general existential results about factorization such as [19, Thm 7.3]. In our case, the situation is additionally exacerbated by the type of symbol of the kernel $\hat{p}_h$ bringing in another entry of exponential type. Indeed, the performed transformation of the conjugation problem on the real axis into the one on the imaginary axis turns the oscillatory factors $e^{2\pi iak}$ into decaying ones, which can sometimes be beneficial to great extent [52, 53], in our case produces another oscillatory behaviour in the diagonal terms due to the symbol of the kernel. See also [29, 71] for reductions of a finite interval integral equation to a Riemann-Hilbert problem with specific matrix coefficients convenient for theoretical analysis.

The presence of both oscillatory and non-analytic exponentials in the matrix reflects the difficulty of finding an exact solution of the equation (2.1) and suggests that the best one can do is aiming at construction of asymptotic approximations to the solutions. However, even known to us analytical approximation strategies are not directly applicable due to lack of strip of analyticity of the symbol usually assumed in the Wiener-Hopf method [2] or contamination of the exponential type of matrix entries, or presence of zero entries [45].

Therefore, more subtle approximation procedures specifically adapted to the present case have to be developed.

2.3 Approximate solutions

We are going to discuss different strategies for obtaining asymptotic solutions: when $\beta \gg 1$ using (2.12), when $\beta \ll 1$ using (2.13). We leave out from the consideration approaches requiring $h \ll 1$ and $a \gg 1$ in (2.1) separately, which could be constructed in much easier manner, but may break the scaling property (e.g. dependence of λ on

β only). Approximation in the opposite case, i.e. when only $a \ll 1$, is less interesting from both theoretical and practical point of view: the kernel p_h is expanded in Taylor series [57] and thus obtained approximate integral operator is only of a finite rank. Essentially the same procedure is proposed and carried out in [31] yielding polynomial representation of eigenfunctions. Such appoximations are rather straighforward but lack the possibility of generating at once[6] infinitely many solutions that the original problem possesses and therefore will also not be discussed here.

2.3.1 Approximation for $\beta \gg 1$

In this subsection we are going to show how to reduce, in an approximate way, the original integral equation to a differential equation of second order which is, in general, much easier to deal with. Such a reduction itself is known to be possible only for a very narrow class of integral equations kernels and is definitely worth pointing out. We, however, do not aim here at rigorously estimation of quality of the approximation in question, leaving this topic for future work.

We start by noting that, for $x \ll 1$,

$$\operatorname{sech}(x) = \frac{1}{1 + x^2/2} + \mathcal{O}\left(x^4\right),$$

and hence the formulation (2.12) is approximated by

$$\int_{-1}^{1} \operatorname{sech}\left((x-t)\sqrt{2}/\beta\right) \phi(t)\, dt = \pi\beta\lambda\phi(x), \quad x \in (-1,1), \tag{2.31}$$

meaning that we expect solutions of both formulations to be close to each other for large values of β.

Observe that on the left we again have a positive compact self-adjoint operator on $L^2(-1,1)$.

This seemingly more complicated integral operator has an advantage over the original one since it belongs to a family of integral operators that admit a commuting differential operator [43, 74]: eigenfunctions of an integral operator with the kernel[7] $\dfrac{b \sin cx}{c \sinh bx}$ (with constants $b, c \in \mathbb{R} \cup i\mathbb{R}$) are also eigenfunctions of the differential operator $-\dfrac{d}{dx}\left(1 - \dfrac{\sinh^2(bx)}{\sinh^2 b}\right)\dfrac{d}{dx} + \left(b^2 + c^2\right)\dfrac{\sinh^2(bx)}{\sinh^2 b}$ with condition of finiteness at $x = \pm 1$. Therefore, taking $c = i\sqrt{2}/\beta$, $b = 2\sqrt{2}/\beta$, and denoting $\dfrac{\mu}{2\sinh^2\left(2\sqrt{2}/\beta\right)}$ an eigenvalue of the differential operator, we reduce (2.31) to solving a boundary value problem for ODE, for $x \in (-1,1)$,

$$\left(\left(\cosh\frac{4\sqrt{2}}{\beta} - \cosh\frac{4\sqrt{2}x}{\beta}\right)\phi'(x)\right)' + \left(\mu - \frac{6}{\beta^2}\left(\cosh\frac{4\sqrt{2}x}{\beta} - 1\right)\right)\phi(x) = 0 \tag{2.32}$$

with boundary conditions

$$\phi'(\pm 1) = \mp\frac{\mu + 6/\beta^2\left(1 - \cosh\left(4\sqrt{2}/\beta\right)\right)}{4\sqrt{2}\beta\sinh\left(4\sqrt{2}/\beta\right)}\phi(\pm 1). \tag{2.33}$$

Alternatively, by change of variable, (2.31) can be brought to the simpler integral equation arising in context

[6]That is, without necessity to increase approximation order to obtain higher order eigenfunctions.

[7]Moreover, this is the only family of kernels sufficiently smooth at the origin which produce integral operators on a symmetric interval admitting commutation with a differential operator of the second order [76].

of singular-value analysis of the finite Laplace transform [5, 21]

$$\int_0^1 \frac{\psi(t)}{s+t+\gamma}dt = -\pi\sqrt{2}\lambda\psi(s), \quad s\in(0,1), \tag{2.34}$$

where $\psi(s) := \dfrac{\phi\left(\dfrac{\beta}{2\sqrt{2}}\log\left[\left(e^{-2\sqrt{2}/\beta}-e^{2\sqrt{2}/\beta}\right)s+e^{-2\sqrt{2}/\beta}\right]\right)}{\left[\left(e^{-2\sqrt{2}/\beta}-e^{2\sqrt{2}/\beta}\right)s+e^{-2\sqrt{2}/\beta}\right]^{1/2}}$, $\gamma := 2e^{-2\sqrt{2}/\beta}$.

The operator in the left-hand side of (2.34) is the finite Stieltjes transform which again, by commutation with a differential operator, can be reduced to solving an ODE, for $s\in(0,1)$,

$$\left(s(1-s)(\gamma+s)(\gamma+1+s)\psi'(s)\right)' - \left(2s(s+\gamma)+\mu\right)\psi(s) = 0 \tag{2.35}$$

with boundary conditions

$$\psi'(0) = \frac{\mu}{\gamma(\gamma+1)}\psi(0), \quad \psi'(1) = -\frac{2(\gamma+1)+\mu}{(\gamma+1)(\gamma+2)}\psi(1) \tag{2.36}$$

imposing regularity of the solution at the endpoints.

It is remarkable that if we get back to (2.32) and expand hyperbolic cosine functions, we obtain

$$\left(\left(1-x^2\right)\phi'(x)\right)' + \left(\mu - \frac{6}{\beta^2}x^2\right)\phi(x) = 0, \quad x\in(-1,1). \tag{2.37}$$

This ODE coincides with the well-studied equation for prolate spheroidal harmonics [51, 63] whose solutions are bounded on $[-1,1]$ only for special values of $\mu_n = \chi_n\left(\dfrac{\sqrt{6}}{\beta}\right)$, $n=\mathbb{N}_0$ and termed as spheroidal wave functions $S_{0n}\left(\dfrac{\sqrt{6}}{\beta}, x\right)$ (χ_n, S_{0n} are as defined in [63]). Equation (2.37) is equivalent (again by commutation of the differential and integral operators and simplicity of their spectra) to the convolutional integral equation on $(-1,1)$ with $\dfrac{\sin\left(x\sqrt{6}/\beta\right)}{\pi x}$ kernel, even though the fact of closedness of this equation to (2.12) is not evident directly.

We note that upon further neglection of the last term in (2.37), we get Legendre differential equation whose only bounded solutions are Legendre polynomials $P_n(x)$, the corresponding eigenvalues of the differential operator are $\mu_n = n(n+1)$. However, such a crude approximation is generally not expected to be good for lower order eigenfunctions: a constant solution $P_0(x)$, clearly, does not satisfy (2.37) which is a direct consequence of the fact that the neglected term $\dfrac{6}{\beta^2}x^2$ in the equation could not be dropped when $\mu=0$.

Finally, we notice that approximation of (2.1) with (2.31) is not analytically meaningless for, due to the powerful result of Widom [74], both problems have very close asymptotic distribution of eigenvalues of higher order. It is also worth noting that such an approximation corresponds to replacing a cut along the imaginary axis in the Fourier domain (recall Riemann-Hilbert formulations in Section 2.2) with a densely spaced set of poles.

2.3.2 Approximation for $\beta \ll 1$

Construction of this approximation is a delicate matter. In many works on (non-homogeneous) Love and Lieb-Liniger equations mentioned in the Introduction, it was crucial for asymptotic analysis that $\lambda = 1$, since near-identity operator perturbation was used. This makes those methods inapplicable in the present case whereas techniques viable for situations with $\lambda = -1$ have more potential to be extended for the purpose of spectral analysis. In particular, in [20], without giving any insight and much details, Griffiths used an interesting approach of reduction of a finite interval convolution equation to a problem on half-line. This was achieved by applying to the original equation the resolvent operator corresponding to the equation on the whole line. The resulting problem on a half-line had two kernels depending on sum and difference of arguments, and neglecting the sum kernel, this problem was then solved numerically. It turns out that, even though this approach also fails, its essence can be transfered to treat a homogeneous version of the equation with λ outside the resolvent set. Analysis becomes much more complicated and requires other observations about problem extension off the interval to be made to remedy the situation with poor behaviour at infinity. Eventually, the interval problem is transformed into the formulation on two symmetric rays, that is, by the solution parity, on a single half-line. The half-line problem, which becomes an integro-differential equation (rather than an integral half-line equation, as in [20]), is approximately solved (neglecting the sum kernel) by use of a Wiener-Hopf type of method, and the solution is, then, analytically continued back to the interval. A constraint requiring solution continuity through the boundary points of the interval results in the characteristic equation for λ.

Denote $B := (-1/\beta, 1/\beta)$ and, similarly to (2.18), we define the analytic continuation off $B \times \{0\}$ to $\mathbb{C} \backslash \{z = x \pm i, x \in B\}$ of the solution of (2.13) as

$$\varphi(z) = \frac{1}{\lambda\pi} \int_{-1/\beta}^{1/\beta} \frac{\varphi(t)}{(z-t)^2 + 1} dt. \tag{2.38}$$

Lemma 2.3.1. *The analytic continuation of the solution of (2.13) given by (2.38) satisfies*

$$\int_{\mathbb{R}\backslash B} R_0(z-t)\,\varphi(t)\,dt = \varphi(z), \quad z = x + iy, \quad |y| < 1, \tag{2.39}$$

with either

$$R_0(z) := \fint_{\mathbb{R}} \frac{e^{-2\pi i k(x+iy)}}{1 - \lambda e^{2\pi|k|}} dk = \fint_{\mathbb{R}} \frac{e^{-ik(x+iy)}}{1 - \lambda e^{|k|}} dk, \tag{2.40}$$

and so (2.39) holds true for any finite x, or, alternatively, for all $x \in \mathbb{R}$, with

$$R_0(z) \quad := \quad S_0(z) + \mathcal{T}(z) + W(z), \tag{2.41}$$

where

$$S_0(z) := -\mathrm{sgn} x \sin((x+iy)\log\lambda), \tag{2.42}$$

$$\mathcal{T}(z) := -2\mathrm{sgn} x \frac{\sin((x+iy)\log\lambda)}{e^{2\pi\ \mathrm{sgn} x(x+iy)} - 1}, \tag{2.43}$$

$$W(z) := -\frac{\lambda}{\pi}\int_0^{\infty}\frac{e^{-\operatorname{sgn}x(x+iy)t}\sin t}{(1-\lambda\cos t)^2+\lambda^2\sin^2 t}dt \tag{2.44}$$

$$= -\frac{\lambda}{\pi}\frac{1}{1-e^{-2\pi\operatorname{sgn}x(x+iy)}}\int_0^{2\pi}\frac{e^{-\operatorname{sgn}x(x+iy)t}\sin t}{1-2\lambda\cos t+\lambda^2}dt \tag{2.45}$$

$$= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\lambda^n}{n^2+(x+iy)^2}. \tag{2.46}$$

Proof. Denote

$$k_0 := -\frac{\log\lambda}{2\pi} > 0 \tag{2.47}$$

with log meaning the principal branch of the logarithmic function so that $\log\lambda \in \mathbb{R}_-$ for $\lambda \in (0,1)$, and let us define, for some constant $0 < \delta < k_0$,

$$R_\delta(x) := \int_{\mathbb{R}\setminus U_\delta}\frac{e^{-2\pi ikx}}{1-\lambda e^{2\pi|k|}}dk, \tag{2.48}$$

$$U_\delta := (-k_0-\delta, -k_0+\delta)\cup(k_0-\delta, k_0+\delta) =: U_{\delta_-}\cup U_{\delta_+}.$$

By symmetry of the integration region, R_δ is even and real-valued, and so is its Fourier transform

$$\hat{R}_\delta(k) = \begin{cases}\dfrac{1}{1-\lambda e^{2\pi|k|}}, & k\notin U_\delta,\\[2mm] 0, & k\in U_\delta.\end{cases} \tag{2.49}$$

Noting that, for $k\notin U_\delta$,

$$\hat{p}_1(k)\hat{R}_\delta(k) = \frac{e^{-2\pi|k|}}{1-\lambda e^{2\pi|k|}} = \hat{p}_1(k)+\lambda\hat{R}_\delta(k),$$

and thus, for $k\in\mathbb{R}$,

$$\hat{p}_1(k)\hat{R}_\delta(k) = \chi_{\mathbb{R}\setminus U_\delta}(k)\hat{p}_1(k)+\lambda\hat{R}_\delta(k).$$

We take inverse Fourier transform of both sides to arrive at

$$\int_{\mathbb{R}}p_1(x-t)R_\delta(t)\,dt = p_1(x)+\lambda R_\delta(x)-\int_{U_\delta}e^{-2\pi ixk}e^{-2\pi|k|}dk,\quad x\in\mathbb{R}. \tag{2.50}$$

Now, extending φ as in (2.38), we convolve with R_δ the equation (2.13) which is now valid on $\mathbb{R}$, and use the relation (2.50)

$$\int_B\int_{\mathbb{R}}p_1(x-t)R_\delta(x-y)\,dx\varphi(t)\,dt = \lambda\int_{\mathbb{R}}\varphi(x)R_\delta(x-y)\,dx,\quad y\in\mathbb{R},$$

$$\Rightarrow\quad \int_B p_1(y-t)\varphi(t)\,dt-\int_B\int_{U_\delta}e^{-2\pi i(y-t)k}e^{-2\pi|k|}dk\varphi(t)\,dt = \lambda\int_{\mathbb{R}\setminus B}R_\delta(y-t)\varphi(t)\,dt,$$

$$\Rightarrow\quad \int_{\mathbb{R}\setminus B}R_\delta(x-t)\varphi(t)\,dt = \varphi(x)-\frac{1}{\lambda}\int_B\int_{U_\delta}e^{-2\pi i(x-t)k}e^{-2\pi|k|}dk\varphi(t)\,dt,\quad x\in\mathbb{R}, \tag{2.51}$$

where the interchange of integration signs is justified by Fubini theorem due to regularity of R_δ: from (2.49), $\hat{R}_\delta \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, and thus the isometry property of Fourier transform and Riemann-Lebesgue lemma imply that $R_\delta \in C(\mathbb{R})\cap L^2(\mathbb{R})$.

Now let us pass to the limit as $\delta\to 0$ in (2.51) to obtain (2.39).

First of all, it is immediate that

$$\left| \int_B \int_{U_\delta} e^{-2\pi i (x-t)k} e^{-2\pi |k|} dk \, \varphi(t) \, dt \right| \leq 4\delta \, \|\varphi\|_{L^1(B)},$$

and it remains to show that, for finite x,

$$\lim_{\delta \searrow 0} \int_{\mathbb{R} \setminus B} \left[R_0 (x-t) - R_\delta (x-t) \right] \varphi(t) \, dt = 0, \tag{2.52}$$

where R_0 is defined by (2.40).

By symmetry of the integration region, we have

$$\begin{aligned} R_0(x) - R_\delta(x) &= 2 \int_{U_{\delta+}} \frac{\cos(2\pi k x) - \cos(2\pi k_0 x)}{1 - \lambda e^{2\pi k}} dk + 2\cos(2\pi k_0 x) \fint_{U_{\delta+}} \frac{dk}{1 - \lambda e^{2\pi k}} \\ &= 2 \int_{-\delta}^{\delta} \frac{\cos(2\pi (k_0 + \theta) x) - \cos(2\pi k_0 x)}{1 - e^{2\pi\theta}} d\theta + 2\delta \cos(2\pi k_0 x) \end{aligned}$$

since, due to direct evaluation,

$$\fint_{U_{\delta+}} \frac{dk}{1 - \lambda e^{2\pi k}} = \fint_{-\delta}^{\delta} \frac{d\theta}{1 - e^{2\pi\theta}} = 2\delta + \frac{1}{2\pi} \lim_{\epsilon \searrow 0} \left(\int_{-2\pi\delta}^{-2\pi\epsilon} + \int_{2\pi\epsilon}^{2\pi\delta} \right) \frac{e^t \, dt}{1 - e^t} = \delta.$$

Now, defining $\varphi_0(x) := \chi_{\mathbb{R}\setminus B}(x) \varphi(x)$, we elaborate

$$\begin{aligned} \int_{\mathbb{R}} \left[R_0(x-t) - R_\delta(x-t) \right] \varphi_0(t) \, dt &= 2\delta \int_{\mathbb{R}} \cos(2\pi k_0 (x-t)) \varphi_0(t) \, dt + \int_{-\delta}^{\delta} \left[e^{2\pi i k_0 x} \left(e^{2\pi i x \theta} \hat{\varphi}_0(-k_0 - \theta) - \hat{\varphi}_0(-k_0) \right) \right. \\ &\quad \left. + e^{-2\pi i k_0 x} \left(e^{-2\pi i x\theta} \hat{\varphi}_0(k_0 + \theta) - \hat{\varphi}_0(k_0) \right) \right] \frac{d\theta}{1 - e^{2\pi\theta}} \\ &= 2\delta \int_{\mathbb{R}} \cos(2\pi k_0 (x-t)) \varphi_0(t) \, dt + e^{2\pi i k_0 x} \int_{-\delta}^{\delta} \frac{e^{2\pi i x\theta} \left[\hat{\varphi}_0(-k_0 - \theta) - \hat{\varphi}_0(-k_0) \right]}{1 - e^{2\pi\theta}} d\theta \\ &\quad + e^{-2\pi i k_0 x} \int_{-\delta}^{\delta} \frac{e^{-2\pi i x\theta} \left[\hat{\varphi}_0(k_0 + \theta) - \hat{\varphi}_0(k_0) \right]}{1 - e^{2\pi\theta}} d\theta \\ &\quad + e^{2\pi i k_0 x} \hat{\varphi}_0(-k_0) \int_{-\delta}^{\delta} \frac{e^{2\pi i x\theta} - 1}{1 - e^{2\pi\theta}} d\theta + e^{-2\pi i k_0 x} \hat{\varphi}_0(k_0) \int_{-\delta}^{\delta} \frac{e^{-2\pi i x\theta} - 1}{1 - e^{2\pi\theta}} d\theta. \end{aligned}$$

Since $\varphi_0 \in L^1(\mathbb{R})$, Riemann-Lebesgue lemma ensures the continuity of $\hat{\varphi}_0$, i.e. $|\hat{\varphi}_0(k_0 + \theta) - \hat{\varphi}_0(k_0)| \leq C|\theta|$, $|\hat{\varphi}_0(-k_0 - \theta) - \hat{\varphi}_0(-k_0)| \leq C|\theta|$, for some $C > 0$, and thus the first three integral terms are small uniformly in $x \in \mathbb{R}$ for small δ. Therefore, we focus on the last two integral terms which can be combined together using that $\hat{\varphi}_0(-k_0) = \overline{\hat{\varphi}_0(k_0)}$ (since φ_0 is real-valued) and thus written as

$$2\mathrm{Re} \left[e^{-2\pi i k_0 x} \hat{\varphi}_0(k_0) \right] \int_{-\delta}^{\delta} \frac{\cos(2\pi x\theta) - 1}{1 - e^{2\pi\theta}} d\theta + 2\mathrm{Im} \left[e^{-2\pi i k_0 x} \hat{\varphi}_0(k_0) \right] \int_{-\delta}^{\delta} \frac{\sin(2\pi x\theta)}{1 - e^{2\pi\theta}} d\theta$$

$$= 2\mathrm{Re} \left[e^{-2\pi i k_0 x} \hat{\varphi}_0(k_0) \right] \int_0^{\delta} \left[\cos(2\pi x\theta) - 1 \right] d\theta + 2\mathrm{Im} \left[e^{-2\pi i k_0 x} \hat{\varphi}_0(k_0) \right] \int_0^{\delta} \frac{\sin(2\pi x\theta) \sinh(2\pi\theta)}{1 - \cosh(2\pi\theta)} d\theta$$

$$= 2\mathrm{Re} \left[e^{-2\pi i k_0 x} \hat{\varphi}_0(k_0) \right] \left(\frac{\sin(2\pi x\delta)}{2\pi x} - \delta \right) + \frac{1}{\pi} \mathrm{Im} \left[e^{-2\pi i k_0 x} \hat{\varphi}_0(k_0) \right] \int_0^{2\pi\delta} \frac{\sin(x\theta) \sinh\theta}{1 - \cosh\theta} d\theta$$

The first term here is uniformly small in $x \in \mathbb{R}$ for small δ while the integral in the second one requires further transformation. To proceed with this, let us introduce the sine integral function

$$\mathrm{Si}\,(x) := \int_0^x \frac{\sin t}{t} dt = \int_0^1 \frac{\sin (xt)}{t} dt.$$

Then, integration by parts gives

$$\int_0^{2\pi\delta} \frac{\sin (x\theta) \sinh \theta}{1-\cosh \theta} d\theta = \int_0^{2\pi\delta} \frac{\sin (x\theta)}{\theta} \frac{\theta \sinh \theta}{1-\cosh \theta} d\theta = \mathrm{Si}\,(2\pi\delta x) \frac{2\pi\delta \sinh (2\pi\delta)}{1-\cosh (2\pi\delta)} - \int_0^{2\pi\delta} \mathrm{Si}\,(x\theta) \frac{\sinh \theta - \theta}{1-\cosh \theta} d\theta.$$

Now since $\mathrm{Si}\,(x\theta) > 0$ for $x,\, \theta > 0$ and $\left(\frac{\sinh \theta - \theta}{1-\cosh \theta}\right)$ is a continuous function on $[0, 2\pi\delta]$, we can apply integral mean value theorem in the last integral

$$\int_0^{2\pi\delta} \mathrm{Si}\,(x\theta) \frac{\sinh \theta - \theta}{1-\cosh \theta} d\theta = \frac{\sinh \theta_0 - \theta_0}{1-\cosh \theta_0} \int_0^{2\pi\delta} \mathrm{Si}\,(x\theta)\, d\theta = \frac{\sinh \theta_0 - \theta_0}{1-\cosh \theta_0} \left[2\pi\delta \mathrm{Si}\,(2\pi\delta x) - \frac{1}{x} (\cos (2\pi\delta x) - 1)\right]$$

for some $\theta_0 = \theta_0\,(x) \in [0, 2\pi\delta]$.

Due to boundedness of the expression in square brackets (observe that $|\mathrm{Si}\,(2\pi x\delta)| \leq \pi/2$),

$$\int_0^{2\pi\delta} \mathrm{Si}\,(x\theta) \frac{\sinh \theta - \theta}{1-\cosh \theta} d\theta = \mathcal{O}\,(\delta)\,, \quad \delta \ll 1$$

uniformly in $x \in \mathbb{R}$.

Hence, we conclude that non-uniformity in x in the limit in (2.52) comes from the single term

$$\frac{1}{\pi} \mathrm{Im}\left[e^{-2\pi i k_0 x} \hat{\varphi}_0\,(k_0)\right] \mathrm{Si}\,(2\pi\delta x) \frac{2\pi\delta \sinh (2\pi\delta)}{1-\cosh (2\pi\delta)}$$

which goes to zero with δ only for finite values of x. Of course, for finite x, the statement of (2.52) could be shown in much more easy fashion, but the delicate analysis that we performed demonstrates that this assumption on x cannot be dropped in the current representation.

To deduce an alternative, more convenient, representation (2.41), we get back to (2.48) and deform the integration contour to indent singularities on the real axis before passing to the limit as $\delta \to 0$. Namely, we introduce the contours $\mathcal{C}_{\delta_-} := \left\{k \in \mathbb{C} : \quad k = -k_0 + \delta e^{i\theta},\, \theta \in (-\pi, 0)\right\}$, $\mathcal{C}_{\delta_+} := \left\{k \in \mathbb{C} : \quad k = k_0 + \delta e^{i\theta},\, \theta \in (-\pi, 0)\right\}$ oriented in the direction of the increase of the argument θ, and, assuming for the moment $x > 0$, we apply Cauchy integral formula for each of two integrals to the left of the last equality in

$$R_\delta\,(x) = \int_{\mathbb{R}_- \setminus U_{\delta_-}} \frac{e^{-2\pi i k x} dk}{1-\lambda e^{-2\pi k}} + \int_{\mathbb{R}_+ \setminus U_{\delta_+}} \frac{e^{-2\pi i k x} dk}{1-\lambda e^{2\pi k}} = \mathcal{T}\,(x) - \left(\int_{\mathcal{C}_{\delta_-}} + \int_0^{-i\infty}\right) \frac{e^{-2\pi i k x} dk}{1-\lambda e^{-2\pi k}} - \left(\int_{\mathcal{C}_{\delta_+}} + \int_{-i\infty}^{0}\right) \frac{e^{-2\pi i k x} dk}{1-\lambda e^{2\pi k}},$$

where we closed the integration contours using decay at infinity of integrands in the lower quadrants of the complex plane and included residue contributions from infinite number of simple poles in the term

$$
\begin{aligned}
\mathcal{T}(x) &= -2\pi i \sum_{n=1}^{\infty} \left[\operatorname{Res}\left(\frac{e^{-2\pi i k x}}{1-\lambda e^{-2\pi k}},\, k=-k_0-in \right) + \operatorname{Res}\left(\frac{e^{-2\pi i k x}}{1-\lambda e^{2\pi k}},\, k=k_0-in \right) \right] \\
&= -i\sum_{n=1}^{\infty} e^{-2\pi n x}\left(e^{2\pi i k_0 x}-e^{-2\pi i k_0 x}\right) = 2\sin(2\pi k_0 x)\left(\frac{1}{1-e^{-2\pi x}}-1\right) = \frac{2\sin(2\pi k_0 x)}{e^{2\pi x}-1}.
\end{aligned}
$$

Consider now

$$
\begin{aligned}
\int_{\mathcal{C}_{\delta_-}} \frac{e^{-2\pi i k x}dk}{1-\lambda e^{-2\pi k}} + \int_{\mathcal{C}_{\delta_+}} \frac{e^{-2\pi i k x}dk}{1-\lambda e^{2\pi k}} &= e^{2\pi i k_0 x} i\delta \int_{-\pi}^{0} \frac{e^{-2\pi i\delta x \exp(i\theta)}e^{i\theta}}{1-e^{-2\pi\delta\exp(i\theta)}}d\theta + e^{-2\pi i k_0 x} i\delta \int_{-\pi}^{0} \frac{e^{-2\pi i\delta x \exp(i\theta)}e^{i\theta}}{1-e^{2\pi\delta\exp(i\theta)}}d\theta \\
&= \delta\sin(2\pi k_0 x)\int_{-\pi}^{0} e^{-2\pi i\delta x\exp(i\theta)}e^{i\theta}\left(\frac{1}{1-e^{2\pi\delta\exp(i\theta)}}-\frac{1}{1-e^{-2\pi\delta\exp(i\theta)}}\right)d\theta \\
&\quad + i\delta\cos(2\pi k_0 x)\int_{-\pi}^{0} e^{-2\pi i\delta x\exp(i\theta)}e^{i\theta}\left(\frac{1}{1-e^{2\pi\delta\exp(i\theta)}}+\frac{1}{1-e^{-2\pi\delta\exp(i\theta)}}\right)d\theta.
\end{aligned}
$$

The integral in the last term can be explicitly calculated

$$
\begin{aligned}
\int_{-\pi}^{0} e^{-2\pi i\delta x\exp(i\theta)}e^{i\theta}\left(\frac{1}{1-e^{2\pi\delta\exp(i\theta)}}+\frac{1}{1-e^{-2\pi\delta\exp(i\theta)}}\right)d\theta &= \int_{-\pi}^{0} e^{-2\pi i\delta x\exp(i\theta)}e^{i\theta}d\theta = -i\int_{-1}^{1} e^{-2\pi i\delta x t}dt \\
&= -i\frac{\sin(2\pi\delta x)}{\pi\delta x},
\end{aligned}
$$

while the first term can be elaborated as

$$
\delta\sin(2\pi k_0 x)\int_{-\pi}^{0} e^{-2\pi i\delta x\exp(i\theta)}e^{i\theta}\frac{e^{2\pi\delta\exp(i\theta)}-e^{-2\pi\delta\exp(i\theta)}}{2-e^{2\pi\delta\exp(i\theta)}-e^{-2\pi\delta\exp(i\theta)}}d\theta
$$

$$
= -\frac{1}{\pi}\sin(2\pi k_0 x)\int_{-\pi}^{0} e^{-2\pi i\delta x\exp(i\theta)}\left(1+r_\delta(\theta)\right)d\theta,
$$

where the $r_\delta(\theta) = \mathcal{O}(\delta^2)$ term was obtained by expanding the exponential terms in powers of δ in both the numerator and the denominator, it is independent of x and uniformly small with δ for any $\theta \in [-\pi, 0]$.

Using the change of variable $\phi = \theta + \pi/2$, we can see real-valuedness of the integral

$$
\int_{-\pi}^{0} e^{-2\pi i\delta x\exp(i\theta)}d\theta = 2\int_{0}^{\pi/2} e^{-2\pi\delta x\cos\phi}\cos(2\pi\delta x\sin\phi)\,d\phi.
$$

Therefore, denoting

$$
\begin{aligned}
S_\delta(x) &:= -\int_{\mathcal{C}_{\delta_-}} \frac{e^{-2\pi i k x}dk}{1-\lambda e^{-2\pi k}} - \int_{\mathcal{C}_{\delta_+}} \frac{e^{-2\pi i k x}dk}{1-\lambda e^{2\pi k}} \\
&= -\cos(2\pi k_0 x)\frac{\sin(2\pi\delta x)}{\pi x} + \frac{2}{\pi}\sin(2\pi k_0 x)\int_{0}^{\pi/2} e^{-2\pi\delta x\cos\phi}\cos(2\pi\delta x\sin\phi)\,d\phi \\
&\quad + \frac{1}{\pi}\sin(2\pi k_0 x)\int_{-\pi}^{0} e^{2\pi\delta x(\sin\theta - i\cos\theta)}r_\delta(\theta)\,d\theta, \qquad (2.53)
\end{aligned}
$$

we arrive at

$$R_\delta(x) = \mathcal{T}(x) + S_\delta(x) - 2\lambda \int_0^\infty \frac{e^{-2\pi x t}\sin(2\pi t)}{[1-\lambda\cos(2\pi t)]^2 + \sin^2(2\pi t)} dt, \quad x > 0. \tag{2.54}$$

From (2.48), it is known that R_δ is an even function, thus extension of the representation (2.54) to negative values is furnished simply by replacing all occurrences of x with $|x|$. Alternatively, one can readily repeat the computations of $R_\delta(x)$ for $x < 0$ initially deforming the contours in quadrants of the upper rather than lower complex half-plane. Since $\hat{R}_\delta \in L^1(\mathbb{R})$ (recall (2.49)), Riemann-Lebesgue lemma ensures the continuity at $x = 0$.

Since $\sin\theta \le 0$ for $\theta \in [-\pi,\pi]$ and $\cos\phi \ge 0$ for $\phi \in [-\pi/2,\pi/2]$, the real exponential factors in (2.53) cannot exceed 1, and hence, because also $\dfrac{\sin x}{x} \le 1$, we have, for any $x \in \mathbb{R}$,

$$|S_\delta(x)| \le 1 + 2\delta + \mathcal{O}\left(\delta^2\right). \tag{2.55}$$

This bound along with the absolute integrability of φ justifies the use of dominated convergence theorem to pass to the limit in

$$\lim_{\delta\to 0}\int_{\mathbb{R}\setminus B} S_\delta(x-t)\varphi(t)\,dt = \int_{\mathbb{R}\setminus B} S_0(x-t)\varphi(t)\,dt,$$

where S_0 is a weak type of limit of S_δ computed from (2.53) letting $\delta \to 0$ for fixed $x \in \mathbb{R}$

$$S_0(x) \quad = \quad \sin(2\pi k_0|x|) = -\mathrm{sgn}x\,\sin(x\log\lambda). \tag{2.56}$$

Scaling change of variable in the second integral in (2.54) proves (2.41) with W as in (2.44).

Note that despite the fact that one cannot pass to the limit as $x \to 0$ under the integral sign in (2.44), the value $W(0) = R_0(0) - S_0(0) - \mathcal{T}(0) = R_0(0) + 2k_0$ is well-defined (by continuity of R_0 implied by the exponential decay of $\hat{R}_0$) and should be understood as the limit of integrals as $x \to 0$.

Now we are going to demonstrate equivalent representations (2.45)-(2.46), the latter has the advantage that it gives the value at $x = 0$ by direct evaluation (namely, $W(0) = -\frac{1}{\pi}\sum_{n=1}^\infty \frac{\lambda^n}{n} = \frac{1}{\pi}\log(1-\lambda)$).

By periodicity of trigonometric functions and summation of geometric series, integration range can be reduced to the finite one

$$\begin{aligned}
\int_0^\infty \frac{e^{-|x|t}\sin t}{(1-\lambda\cos t)^2 + \lambda^2\sin^2 t}dt \quad &= \quad \sum_{n=0}^\infty \int_{2\pi n}^{2\pi(n+1)} \cdots = \sum_{n=0}^\infty e^{-2\pi|x|n}\int_0^{2\pi} \frac{e^{-|x|t}\sin t}{(1-\lambda\cos t)^2 + \lambda^2\sin^2 t}dt \\
&= \quad \frac{1}{1-e^{-2\pi|x|}}\int_0^{2\pi} \frac{e^{-|x|t}\sin t}{(1-\lambda\cos t)^2 + \lambda^2\sin^2 t}dt,
\end{aligned} \tag{2.57}$$

which shows (2.45). Here again evaluation at $x = 0$ should be done with care since the pre-integral factor blows up. However, this blow-up is compensated by vanishing of the integral, hence taking limit of the product yields a well-defined value which will become evident after applying further transformations.

We start by noticing that the quantity

$$\frac{1}{(1-\lambda\cos t)^2 + \lambda^2\sin^2 t} = \frac{1}{1-2\lambda\cos t + \lambda^2}$$

is a generating function for Chebyshev polynomials of the second kind U_n [50, (18.12.10)], that is

$$\frac{1}{1-2\lambda\cos t+\lambda^2}=\sum_{n=0}^{\infty}\lambda^n U_n\left(\cos t\right).$$

Since $\lambda<1$, convergence of the series is absolute and uniform in $t\in[0,2\pi]$, and hence it can be integrated termwise.

Integrating by parts,

$$\int_0^{2\pi}\frac{e^{-|x|t}\sin t}{1-2\lambda\cos t+\lambda^2}dt=\frac{1}{|x|}\sum_{n=0}^{\infty}\lambda^n\int_0^{2\pi}e^{-|x|t}\frac{d}{dt}\left(\sin t\,U_n\left(\cos t\right)\right)dt.$$

Employing formula [50, (18.9.21)] $U_n=\dfrac{1}{n+1}T'_{n+1}$ establishing the connection with Chebyshev polynomials of the first kind T_n, we simplify

$$\begin{aligned}\frac{d}{dt}\left(\sin t\,U_n\left(\cos t\right)\right) &= -\sin^2 t\,U'_n\left(\cos t\right)+\cos t\,U_n\left(\cos t\right)\\ &= \frac{1}{n+1}\left[\left(\cos^2 t-1\right)\frac{d^2T_{n+1}\left(\cos t\right)}{d\left(\cos t\right)^2}+\cos t\frac{dT_{n+1}\left(\cos t\right)}{d\left(\cos t\right)}\right]\\ &= \left(n+1\right)T_{n+1}\left(\cos t\right),\end{aligned}$$

where the last equality is due to a differential equation [50, Sect. 18.8] satisfied by T_{n+1}.

Now, since $T_n\left(\xi\right)=\cos\left(n\arccos\xi\right)$ [50, (18.5.1)],

$$\begin{aligned}\int_0^{2\pi}\frac{e^{-|x|t}\sin t}{1-2\lambda\cos t+\lambda^2}dt &= \frac{1}{|x|}\sum_{n=0}^{\infty}\lambda^n\left(n+1\right)\int_0^{2\pi}e^{-|x|t}\cos\left(\left(n+1\right)t\right)dt\\ &= \left(1-e^{-2\pi|x|}\right)\sum_{n=0}^{\infty}\lambda^n\frac{n+1}{x^2+\left(n+1\right)^2},\end{aligned}\tag{2.58}$$

where the integral calculation became elementary

$$\begin{aligned}\int_0^{2\pi}e^{-|x|t}\cos\left(\left(n+1\right)t\right)dt &= \mathrm{Re}\int_0^{2\pi}e^{[i(n+1)-|x|]t}dt\\ &= \mathrm{Re}\left[\frac{e^{-2\pi|x|}-1}{i\left(n+1\right)-|x|}\right]=\frac{|x|\left(1-e^{-2\pi|x|}\right)}{x^2+\left(n+1\right)^2}.\end{aligned}$$

Plugging (2.58) into (2.57) yields (2.46).

Extension of these results off the real axis is straightforward - exactly the same steps can be repeated replacing x with $x+iy$ in both cases $x>0$ and $x<0$ as long as $|y|<1$ so that the convergence of all integrals starting from (2.48) is ensured. □

The kernel $R_0\left(x\right)$ in the half-line reformulation given by the Lemma has a term $S_0\left(x\right)$ which is rather unpleasant for it does not decay at infinity, the fact that prevents us from performing approximation to construct asymptotic solution to the problem.

To remedy the situation, let us observe that, for real-valued arguments, we can rewrite

$$S_0(x) + \mathcal{T}(x) = -\sin(|x|\log\lambda)\left(1 + \frac{2}{e^{2\pi|x|}-1}\right) = -\frac{\sin(x\log\lambda)}{\tanh(\pi x)}, \tag{2.59}$$

which is seen to be a real-analytic function. Then, we differentiate equation (2.39) twice and add it to the equation itself multiplied by $\log^2\lambda$ leading to an integro-differential equation

$$\int_{\mathbb{R}\setminus B} \mathcal{K}(x-t)\,\varphi(t)\,dt = \varphi''(x) + \log^2\lambda\,\varphi(x), \quad x\in\mathbb{R}, \tag{2.60}$$

where

$$\begin{aligned}\mathcal{K}(x) \quad &:= \quad -\left(\frac{d^2}{dx^2} + \log^2\lambda\right)\left(\frac{\sin(x\log\lambda)}{\tanh(\pi x)} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\lambda^n}{n^2+x^2}\right)\\ &= \quad 2\pi\frac{\log\lambda\cos(x\log\lambda) - \pi\sin(x\log\lambda)\coth(\pi x)}{\sinh^2(\pi x)}\\ &\quad\quad -\frac{\log^2\lambda}{\pi}\sum_{n=1}^{\infty}\frac{n\lambda^n}{n^2+x^2} + \frac{2}{\pi}\sum_{n=1}^{\infty}n\lambda^n\frac{n^2-3x^2}{(n^2+x^2)^3}.\end{aligned} \tag{2.61}$$

Despite the complicated form of the kernel (2.61) the obtained equation (2.60) is perfectly suitable for constructing approximate solution for $\beta \ll 1$. Indeed, since $\mathcal{K}$ is even and φ is either an even or an odd function (recall Proposition 2.1.1), the problem translates into one of the following integral equations on the positive half-line

$$\int_0^\infty \left[\mathcal{K}(x-t) \pm \mathcal{K}\left(x+t+\frac{2}{\beta}\right)\right]\varphi\left(t+\frac{1}{\beta}\right)dt = \varphi''\left(x+\frac{1}{\beta}\right) + \log^2\lambda\,\varphi\left(x+\frac{1}{\beta}\right), \quad x>0, \tag{2.62}$$

with the upper sign on the left corresponding to the even parity of φ.

This is an analog of Wiener-Hopf integral equation with two kernels (of Toeplitz and Hankel type) which is known to be solvable in a closed form only in some very special cases [4], and hence, does not present a big hope for arriving at an explicit solution. Nevertheless, this form is convenient for the construction of an approximation since we can take advantage of the decay of the kernel at infinity (the oscillatory S_0 term is now absent!), and thus work only with the convolution part of the integral operator. For reasons that will become apparent in the following computations on the Fourier transform side, it is also convenient to calibrate the solution which extends by zero to the negative half-line as a C^1 function. Namely, we introduce

$$\varphi_0(x) := \chi_{\mathbb{R}_+}(x)\left(\varphi\left(x+\frac{1}{\beta}\right) - e^{-x}\left[(1+x)\,\varphi\left(\frac{1}{\beta}\right) + x\varphi'\left(\frac{1}{\beta}\right)\right]\right) =: \chi_{\mathbb{R}_+}(x)\left(\varphi\left(x+\frac{1}{\beta}\right) - m(x)\right), \tag{2.63}$$

so that $\varphi_0(0) = \varphi_0'(0) = 0$ and hence $\varphi_0''(x) = \chi_{\mathbb{R}_+}(x)\left(\varphi''\left(x+\frac{1}{\beta}\right) - m''(x)\right)$.

Therefore, equation (2.62) rewrites as

$$\int_0^\infty \mathcal{K}(x-t)\,\varphi_0(t)\,dt - \varphi_0''(x) - \log^2\lambda\,\varphi_0(x) = \mathcal{M}(x) + \mathcal{R}(x), \quad x>0, \tag{2.64}$$

$$\mathcal{M}(x) := m''(x) + \log^2\lambda\, m(x) - \int_0^\infty \mathcal{K}(x-t)\, m(t)\, dt, \tag{2.65}$$

$$\mathcal{R}(x) := \mp \int_0^\infty \mathcal{K}\left(x + t + \frac{2}{\beta}\right) \varphi\left(t + \frac{1}{\beta}\right) dt. \tag{2.66}$$

It is the estimate for (2.66) that makes this formulation into an approximation strategy.

Indeed, since eigenfunctions are defined up to a multiplicative constant, we can choose normalization such that $\|\varphi\|_{L^1(B)} = 1$. Then, *a priori* bounds on the residue term $\mathcal{R}$ are due to straightforward estimates: for x, $t \in \mathbb{R}_+$ and some constant $c_\lambda > 0$ dependent on λ, we have

$$|\mathcal{K}(x+t+2/\beta)| \le c_\lambda \left[\frac{1}{(x+t+2/\beta)^2} + \frac{1}{(x+t+2/\beta)^4} + e^{-2\pi(x+t+2/\beta)}\right] \tag{2.67}$$

leading to

$$\left|\int_0^\infty \mathcal{K}\left(x+t+\frac{2}{\beta}\right)\varphi\left(t+\frac{1}{\beta}\right)dt\right| \le c_\lambda \left[\frac{1}{(x+2/\beta)^2} + \frac{1}{(x+2/\beta)^4} + e^{-2\pi(x+2/\beta)}\right] \|\varphi\|_{L^1\left(\frac{1}{\beta},\infty\right)},$$

and hence we get, for $1 \le p \le \infty$,

$$\|\mathcal{R}\|_{L^p(\mathbb{R}_+)} \le c_{p,\lambda}\left(\beta^{2-1/p} + \beta^{4-1/p} + e^{-4\pi/\beta}\right)\|\varphi\|_{L^1\left(\frac{1}{\beta},\infty\right)}, \tag{2.68}$$

with some constant $c_{p,\lambda} > 0$ depending on p and λ.

Even though the kernel $\mathcal{K}$ given by (2.61) has an unwieldy form, its Fourier transform which is crucial in what follows can be expressed in terms of elementary functions.

The series part in (2.61) can be transformed termwise due to uniform convergence for $0 < \lambda < 1$:

$$\mathcal{F}\left[-\frac{1}{\pi}\left(\frac{d^2}{dx^2} + \log^2\lambda\right)\sum_{n=1}^\infty \frac{n\lambda^n}{n^2+x^2}\right](k) = 4\pi^2\left(k^2 - k_0^2\right)\sum_{n=1}^\infty \lambda^n e^{-2\pi n|k|} = 4\pi^2\left(k^2-k_0^2\right)\frac{e^{-2\pi(|k|+k_0)}}{1-e^{-2\pi(|k|+k_0)}}, \tag{2.69}$$

where k_0 is as defined in (2.47).

To compute the rest, it is convenient[8] first to calculate distributional Fourier transform of $\dfrac{\sin(x\log\lambda)}{\tanh(\pi x)}$. This function does not decay at infinity and its Fourier transform will be non-integrable due to singularities at $k = \pm k_0$. After application of the operator $\left(\dfrac{d^2}{dx^2} + \log^2\lambda\right)$, the oscillatory term will be eliminated, while, on the Fourier transform side, the singularities will be suppresed by the factor $\left(k_0^2 - k^2\right)$. Bearing this in mind, we compute

[8] In the same fashion as presented here, we can perform direct computations of the Fourier transform of the first term in (2.61). No distributional interpretation is needed, but calculations are more heavy in such a case. There is also an alternative approach still appealing to distributions (distributional derivatives) which is justified more rigorously as far as the applicability of Cauchy theorem is concerned. Namely,

$$\begin{aligned}
\mathcal{F}\left[-\frac{\sin(x\log\lambda)}{\tanh(\pi x)}\right](k) &= \left(-\frac{d^2}{dk^2}+1\right)\mathcal{F}\left[-\frac{\sin(x\log\lambda)}{(4\pi^2x^2+1)\tanh(\pi x)}\right](k) \\
&= \left(-\frac{d^2}{dk^2}+1\right)\left[\operatorname{sgn}(k+k_0)\left(\frac{1}{2} - \frac{e^{-|k+k_0|}}{4\tan(1/2)} + \sum_{j=1}^\infty \frac{e^{-2\pi|k+k_0|j}}{1-4\pi^2j^2}\right)\right. \\
&\quad \left. -\operatorname{sgn}(k-k_0)\left(\frac{1}{2} - \frac{e^{-|k-k_0|}}{4\tan(1/2)} + \sum_{j=1}^\infty \frac{e^{-2\pi|k-k_0|j}}{1-4\pi^2j^2}\right)\right].
\end{aligned}$$

$$\mathcal{F}\left[-\frac{\sin\left(x\log\lambda\right)}{\tanh\left(\pi x\right)}\right](k)=\frac{1}{2i}\lim_{\delta\to 0}\left[\int_{\mathbb{R}\setminus U_\delta(0)}\frac{e^{2\pi i(k+k_0)x}}{\tanh\left(\pi x\right)}dx-\int_{\mathbb{R}\setminus U_\delta(0)}\frac{e^{2\pi i(k-k_0)x}}{\tanh\left(\pi x\right)}dx\right], \tag{2.70}$$

where $U_\delta(0) := \{x\in\mathbb{R}:\ |x|<\delta\}$.

Denote the contours

$$C_\delta^+ := \left\{z\in\mathbb{C}:\ z=\delta e^{2\pi i\theta},\ \theta\in(0,\pi)\right\},\quad C_\delta^- := \left\{z\in\mathbb{C}:\ z=\delta e^{2\pi i\theta},\ \theta\in(-\pi,0)\right\},$$

that will be traversed in a direction corresponding to the increase of the argument θ.

Suppose, for a moment, that $k+k_0>0$. Then, by Cauchy theorem, closing the contour in the upper half-plane[9], we have

$$\begin{aligned}\lim_{\delta\to 0}\int_{\mathbb{R}\setminus U_\delta(0)}\frac{e^{2\pi i(k+k_0)x}}{\tanh\left(\pi x\right)}dx &= -\lim_{\delta\to 0}\int_{C_\delta^+}\frac{e^{2\pi i(k+k_0)z}}{\tanh\left(\pi z\right)}dz+2\pi i\sum_{j=1}^{\infty}\operatorname{Res}\left(\frac{e^{2\pi i(k+k_0)z}}{\tanh\left(\pi z\right)},\ z=ij\right)\\ &= i+2i\sum_{j=1}^{\infty}e^{-2\pi(k+k_0)j}=i\left(\frac{2}{1-e^{-2\pi(k+k_0)}}-1\right)=i\frac{1+e^{-2\pi(k+k_0)}}{1-e^{-2\pi(k+k_0)}}\\ &= \frac{i}{\tanh\left(\pi\left(k+k_0\right)\right)}.\end{aligned}$$

Similarly, for $k+k_0<0$, we close the contour in the lower half-plane to obtain

$$\begin{aligned}\lim_{\delta\to 0}\int_{\mathbb{R}\setminus U_\delta(0)}\frac{e^{2\pi i(k+k_0)x}}{\tanh\left(\pi x\right)}dx &= -\lim_{\delta\to 0}\int_{C_\delta^-}\frac{e^{2\pi i(k+k_0)z}}{\tanh\left(\pi z\right)}dz-2\pi i\sum_{j=1}^{\infty}\operatorname{Res}\left(\frac{e^{2\pi i(k+k_0)z}}{\tanh\left(\pi z\right)},\ z=-ij\right)\\ &= -i-2i\sum_{j=1}^{\infty}e^{2\pi(k+k_0)j}=-i\left(\frac{2}{1-e^{2\pi(k+k_0)}}-1\right)\\ &= \frac{i}{\tanh\left(\pi\left(k+k_0\right)\right)}.\end{aligned}$$

Therefore, regardless of the sign of $(k+k_0)$, we have

$$\lim_{\delta\to 0}\int_{\mathbb{R}\setminus U_\delta(0)}\frac{e^{2\pi i(k+k_0)x}}{\tanh\left(\pi x\right)}dx=\frac{i}{\tanh\left(\pi\left(k+k_0\right)\right)},$$

and, repeating exactly the same computation for the term dependent on $(k-k_0)$ in (2.70), we arrive at

$$\mathcal{F}\left[-\frac{\sin\left(x\log\lambda\right)}{\tanh\left(\pi x\right)}\right](k)=\frac{1}{2\tanh\left(\pi\left(k+k_0\right)\right)}-\frac{1}{2\tanh\left(\pi\left(k-k_0\right)\right)}.$$

This result along with (2.69) leads to

$$\hat{\mathcal{K}}(k) = 4\pi^2\left(k_0^2-k^2\right)\left[\frac{1}{2\tanh\left(\pi\left(k+k_0\right)\right)}+\frac{1}{2\tanh\left(\pi\left(k_0-k\right)\right)}-\frac{e^{-2\pi(|k|+k_0)}}{1-e^{-2\pi(|k|+k_0)}}\right],$$

which can also be rewritten simplifying the terms under assumption $k>0$ and then restoring the absolute value

[9] Note that the hyperbolic cotangent is a meromorphic function bounded in the whole complex plane except at poles on the imaginary axis whose contribution for large arguments will be suppresed in the limit by the decay of the exponential multiplier on the imaginary axis.

sign in k due to even parity

$$\hat{\mathcal{K}}(k) = 2\pi^2 \left(k_0^2 - k^2\right)\left[1 + \coth\left(\pi\left(k_0 - |k|\right)\right)\right]. \tag{2.71}$$

In particular, this explicit form allows us to estimate precisely how $\mathcal{K}(x)$ decays at infinity and hence also the residue term in our approximation strategy.

The necessity of this estimate is motivated as follows. According to Proposition 2.1.3, $\lambda \nearrow 1$ as $\beta \searrow 0$, and therefore there is a possible obstruction to convergence of a geometric series: evidently, one cannot pass to the limit termwise in the series $\sum_{n=1}^{\infty} \dfrac{n\lambda^n}{n^2 + x^2}$ which is the one responsible for the leading order behavior at infinity of $\mathcal{K}(x)$ given by (2.61). In other words, we have to quantify how exactly the constant $c_{p,\lambda}$ in (2.68) depends on λ and thus β. In order to obtain such a bound, we take inverse Fourier transform of (2.71), splitting the integration range into positive and negative semi-axes and employing repetitive integration by parts in both integrals to conclude with[10]

$$\begin{aligned}
\mathcal{K}(x) &= -\frac{\pi k_0^2}{x^2 \sinh^2(\pi k_0)} + \frac{1}{2x^2}\int_{\mathbb{R}} e^{-2\pi i k x}\left(\left(k_0^2 - k^2\right)\left[1 + \coth\left(\pi\left(k_0 - |k|\right)\right)\right]\right)'' dk \\
&= -\frac{\pi k_0^2}{x^2 \sinh^2(\pi k_0)} + \mathcal{O}\left(\frac{1}{x^4}\right), \quad x \gg 1.
\end{aligned} \tag{2.72}$$

This furnishes an improved version of the bound (2.68)

$$\|\mathcal{R}\|_{L^p(\mathbb{R}_+)} \lesssim c_p \frac{\log^2 \lambda}{\lambda \sinh^2\left(\dfrac{1}{2}\log\lambda\right)} \beta^{2-1/p} \|\varphi\|_{L^1\left(\frac{1}{\beta},\infty\right)}, \quad \beta \ll 1, \tag{2.73}$$

where $c_p > 0$ is a constant which depends only on $1 \leq p \leq \infty$.

The “exterior” norm $L^1\left(\frac{1}{\beta},\infty\right)$ can be easily expressed in terms of the L^1-norm of the solution inside the interval by means of the integral equation itself:

$$\|\varphi\|_{L^1\left(\frac{1}{\beta},\infty\right)} \leq \|\varphi\|_{L^1(\mathbb{R}_+)} \leq \frac{1}{2\lambda}\|\varphi\|_{L^1(B)},$$

however, due to natural normalization in Hilbertian setting, we prefer to obtain an analog of this in $L^2(B)$.

To get such an estimate, we split the integration range

$$\|\varphi\|_{L^1\left(\frac{1}{\beta},\infty\right)} \leq \frac{1}{\pi\lambda}\left(\int_{\frac{1}{\beta}}^{\frac{1}{\beta}+1} + \int_{\frac{1}{\beta}+1}^{\infty}\right)\int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \frac{|\varphi(t)|\,dt}{(x-t)^2 + 1}dx,$$

and use Cauchy-Schwarz inequality: twice in the first integral and once in the second one. This yields

$$\|\varphi\|_{L^1\left(\frac{1}{\beta},\infty\right)} \leq \frac{\|\varphi\|_{L^2(B)}}{\pi\lambda}\left(\left(\int_{\frac{1}{\beta}}^{\frac{1}{\beta}+1}\int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \frac{dt}{\left((x-t)^2+1\right)^2}dx\right)^{1/2} + \int_{\frac{1}{\beta}+1}^{\infty}\left(\int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \frac{dt}{\left((x-t)^2+1\right)^2}dx\right)^{1/2}\right).$$

[10] Strictly speaking, the $\mathcal{O}$ term here depends on k_0 and hence on λ. However, it can be seen (for example, by continueing integration by parts), that in the both limiting cases of our interest $\lambda \searrow 0$ (e.g. when index of an eigenvalue increases) and $\lambda \nearrow 1$ (when the index is fixed while we take β smaller) it does not blow up.

Since $\left((x-t)^2+1\right)^2 \geq (x-t)^2+1$, in the first integral we bound

$$\int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \frac{dt}{\left((x-t)^2+1\right)^2} \leq \int_{-\infty}^{\infty} \frac{dt}{(x-t)^2+1} = \frac{1}{\pi}.$$

In the second integral, since x is strictly outside B, we can estimate

$$\int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \frac{dt}{\left((x-t)^2+1\right)^2} \leq \int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \frac{dt}{(x-t)^4} \leq \frac{1}{3\,(x-1/\beta)^3},$$

and the overall bound is then

$$\|\varphi\|_{L^1\left(\frac{1}{\beta},\infty\right)} \leq \frac{\|\varphi\|_{L^2(B)}}{\pi\lambda}\left(\frac{2}{\sqrt{3}}+\sqrt{\pi}\right).$$

Hence, assuming the normalization $\|\varphi\|_{L^2(B)}=1$, we have

$$\|\mathcal{R}\|_{L^p(\mathbb{R}_+)} = \mathcal{O}\left(\frac{\log^2\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\beta^{2-1/p}\right), \quad \beta \ll 1. \tag{2.74}$$

To facilitate formulation of further results, it is convenient to introduce projection operators on H_+, H_-, the spaces of functions holomorphic in the upper and the lower complex half-planes, respectively. Let F be a Hölder continuous function decaying at infinity. Then, by Paley-Wiener theorem, the functions

$$P_+\left[F\right](k) := \mathcal{F}\chi_{\mathbb{R}_+}\mathcal{F}^{-1}\left[F\right](k) = \frac{1}{2\pi}\int_0^{\infty} e^{ikx}\int_{\mathbb{R}} e^{-itx}F(t)\,dt\,dx, \tag{2.75}$$

$$P_-\left[F\right](k) := \mathcal{F}\chi_{\mathbb{R}_-}\mathcal{F}^{-1}\left[F\right](k) = \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx}\int_{\mathbb{R}} e^{-itx}F(t)\,dt\,dx, \tag{2.76}$$

realize projections on H_+ and H_-, which also, by Plemelj-Sokhotskii formulas, can be rewritten as

$$P_+\left[F\right](k) = \frac{1}{2}F(k) + \frac{1}{2\pi i}\fint_{\mathbb{R}} \frac{F(t)}{t-k}dt, \tag{2.77}$$

$$P_-\left[F\right](k) = \frac{1}{2}F(k) - \frac{1}{2\pi i}\fint_{\mathbb{R}} \frac{F(t)}{t-k}dt. \tag{2.78}$$

Finally, let us also define few auxiliary quantities, for $k \in \mathbb{R}$,

$$G(k) := \frac{k^2-k_0^2}{2\left(k^2+1\right)}\left[1+\coth\left(\pi\left(|k|-k_0\right)\right)\right], \tag{2.79}$$

$$\kappa := -2\left(\pi + \int_0^{\infty}\log G(k)\,dk\right), \tag{2.80}$$

$$\mathcal{C}(k) := \frac{(1+\kappa)\left(1+4\pi^2k_0^2\right)}{(1-2\pi ik)^2} - \frac{1-4\pi^2k_0^2+2\kappa}{1-2\pi ik} - P_+\left[\frac{2\left(1-\pi i\cdot\right)+\kappa}{\left(1-2\pi i\cdot\right)^2}\hat{\mathcal{K}}(\cdot)\right](k), \tag{2.81}$$

$$C_\lambda := \frac{\log^{1/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\left(1+\log^5\lambda+\frac{1+\log^4\lambda}{\lambda^{1/2}}\right), \tag{2.82}$$

$$C_\lambda^{(0)} := \frac{(1+\log\lambda)\log^{3/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}. \tag{2.83}$$

We are now ready for the following result which provides approximate inversion of the integral operator on the half-line.

Proposition 2.3.1. *For $\beta \ll 1$, the approximate equation (2.64) has the unique $W^{2,2}(\mathbb{R}_+)$ solution given by*

$$\varphi_0(x) \quad = \quad \varphi\left(\frac{1}{\beta}\right)\int_{\mathbb{R}} e^{-2\pi ikx}\frac{P_+[\mathcal{C}/G](k)}{4\pi^2(k^2+1)}dk + \mathcal{E}_0(x), \tag{2.84}$$

and, moreover,

$$\varphi'\left(\frac{1}{\beta}\right) = \kappa\varphi\left(\frac{1}{\beta}\right) + \varepsilon_0. \tag{2.85}$$

The error terms in this approximation can be estimated as $\|\mathcal{E}_0\|_{W^{2,2}(\mathbb{R}_+)} = \mathcal{O}\left(C_\lambda\beta^{3/2}\right)$, $\varepsilon_0 = \mathcal{O}\left(C_\lambda^{(0)}\beta^{3/2}\right)$ provided that $\|\varphi\|_{L^2(B)} = 1$.

Proof. Standard solution procedure is to reduce the integral equation to a conjugation problem for analytic functions in the Fourier domain [14, 47]. To this effect, let us extend the equation to the whole line as

$$\int_{\mathbb{R}}\mathcal{K}(x-t)\varphi_0(t)\,dt - \varphi_0''(x) - \log^2\lambda\,\varphi_0(x) = \psi(x) + \mathcal{M}_0(x) + \mathcal{R}_0(x), \quad x\in\mathbb{R}, \tag{2.86}$$

where

$$\mathcal{M}_0(x) := \chi_{\mathbb{R}_+}(x)\mathcal{M}(x), \quad \mathcal{R}_0(x) := \chi_{\mathbb{R}_+}(x)\mathcal{R}(x),$$

$$\psi(x) := \chi_{\mathbb{R}_-}(x)\int_0^\infty \mathcal{K}(x-t)\varphi_0(t)\,dt.$$

Taking Fourier transform of (2.86), we obtain

$$G(k)\Phi_+(k) - \Phi_-(k) = \hat{\mathcal{M}}_0(k) + \hat{\mathcal{R}}_0(k), \quad k\in\mathbb{R}, \tag{2.87}$$

where

$$G(k) := \frac{\hat{\mathcal{K}}(k) + 4\pi^2\left(k^2-k_0^2\right)}{4\pi^2(k^2+1)}, \tag{2.88}$$

$$\Phi_+(k) := 4\pi^2\left(k^2+1\right)\hat{\varphi}_0(k), \quad \Phi_-(k) := \hat{\psi}(k), \tag{2.89}$$

and $G(k)$ more explicitly is given by (2.79).

By Paley-Wiener theorem, $\Phi_\pm \in H_\pm$ meaning that Φ_+ and Φ_- are boundary values of functions holomorphic in the upper and the lower half-planes of complex plane, accordingly.

We have thus obtained a scalar Riemann-Hilbert problem that consists in finding sectionally holomorphic functions Φ_+, Φ_- whose restrictions to the real axis $\mathbb{R}$ satisfy the conjugation condition (2.87) with the right-hand side being a given function.

To proceed with solution of (2.87), one has to construct a factorization of the coefficient of the problem $G(k) = X_+(k)\, X_-(k)$, where $X_\pm \in H_\pm$ are zero-free functions in respective half-planes. Unlike in the classical Wiener-Hopf method, instead of a strip of analyticity, $G(k)$ is defined only on the real line $\mathbb{R}$. However, since $G(k)$ is a Hölder continuous function that does not vanish on the whole real line, its factorization can still be constructed in terms of Cauchy principal value integrals [13, 14, 30]. In the present case, $G(k)$ is Lipschitz continuous as is $|k|$, and evidently $G(k) > 0$ on $\mathbb{R}$ for any $k_0 > 0$ (i.e. any $\lambda \in (0,1)$).

The last inequality entails that $\log G(k)$ is well-defined and, because of real-valuedness of $G(k)$ on $\mathbb{R}$ (that implies it has zero index), it is also a single-valued function.

Since $G(\tau) \to 1$ as $\tau \to \pm\infty$, we can define, for $k \notin \mathbb{R}$,

$$\mathcal{G}(k) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log G(\tau)}{\tau - k} d\tau,$$

which is an analytic function in $\mathbb{C}\backslash\mathbb{R}$.

As mentioned, $G(k)$ is Lipschitz continuous on $\mathbb{R}$ and so is $\log G(k)$, hence Plemelj-Sokhotski formulas apply to yield boundary values of this Cauchy integral

$$\mathcal{G}_\pm(k) = \pm\frac{1}{2}\log G(k) + \frac{1}{2\pi i}\fint_{\mathbb{R}} \frac{\log G(\tau)}{\tau - k} d\tau, \quad k \in \mathbb{R},$$

and, in particular,

$$\log G(k) = \mathcal{G}_+(k) - \mathcal{G}_-(k),$$

which furnishes the desired factorization

$$G(k) = \exp \mathcal{G}_+(k) \exp\left(-\mathcal{G}_-(k)\right) =: X_+(k)\, X_-(k), \tag{2.90}$$

where X_+ and X_- are functions holomorphic and zero-free in respective half-planes given by

$$X_\pm(k) := \exp\left(P_\pm[\log G](k)\right) = G^{1/2}(k) \exp\left[\pm\frac{1}{2\pi i}\fint_{\mathbb{R}} \frac{\log G(\tau)}{\tau - k} d\tau\right], \tag{2.91}$$

and the branch of the square root is taken so that it has positive values for positive arguments.

We have, therefore, arrived at

$$X_+(k)\,\Phi_+(k) - \Phi_-(k)/X_-(k) = \left[\hat{\mathcal{M}}_0(k) + \hat{\mathcal{R}}_0(k)\right]/X_-(k). \tag{2.92}$$

Since the last term on the right is Lipschitz continuous and tends to zero at infinity, we can write

$$\left[\hat{\mathcal{M}}_0(k) + \hat{\mathcal{R}}_0(k)\right]/X_-(k) = Y_+(k) - Y_-(k),$$

with Y_+, Y_- being boundary values of the Cauchy integral

$$Y(k) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\left[\hat{\mathcal{M}}_0(\tau) + \hat{\mathcal{R}}_0(\tau)\right]/X_-(\tau)}{\tau - k} d\tau, \quad k \notin \mathbb{R}. \tag{2.93}$$

Rearranging the terms in (2.92), we have

$$X_+(k)\,\Phi_+(k) - Y_+(k) = \Phi_-(k)/X_-(k) - Y_-(k). \tag{2.94}$$

It then follows that the left- and right-hand sides must be a restriction to $\mathbb{R}$ of one entire function. Note that $\hat{\varphi}_0(k) = \mathcal{O}\left(\frac{1}{k^3}\right)$ for large k, as can be seen from repetitive integration by parts of $\int_0^\infty e^{2\pi i k x}\varphi_0(x)\,dx$ taking into account that $\varphi_0(0) = \varphi_0'(0) = 0$ (recall (2.63)). Therefore, $\Phi_+(k) = \mathcal{O}\left(\frac{1}{k}\right)$ and all the terms in (2.94) decay at infinity implying that, by Liouville theorem [1], this entire function must be identically zero.

This reasoning yields

$$\Phi_+(k) = Y_+(k)/X_+(k), \quad k \in \mathbb{R},$$

and hence, recalling (2.89), we obtain

$$\hat{\varphi}_0(k) = \frac{Y_+(k)}{4\pi^2(k^2+1)\,X_+(k)}. \tag{2.95}$$

We observe that the denominator has a zero at $k = i$ whereas we know that $\hat{\varphi}_0 \in H_+$, there must be an additional condition imposed

$$Y(i) = Y_+(i) = 0 \quad \Rightarrow \quad \int_{\mathbb{R}} \frac{\left[\hat{\mathcal{M}}_0(t) + \hat{\mathcal{R}}_0(t)\right]/X_-(t)}{t - i} dt = 0, \tag{2.96}$$

where the last implication is due to (2.93).

Note that writing

$$\frac{k^2+1}{k^2-k_0^2} = \frac{(|k|-k_0)^2 + 2k_0(|k|-k_0) + k_0^2 + 1}{(|k|-k_0)(|k|+k_0)} = 1 + \frac{k_0^2+1}{(|k|-k_0)(|k|+k_0)},$$

$$\frac{1}{1+\coth(\pi(|k|-k_0))} = \sinh(\pi(|k|-k_0))\,e^{-\pi(|k|-k_0)} = \frac{1}{2}\left(1 - e^{-2\pi(|k|-k_0)}\right),$$

and using the inequality

$$\frac{\sinh(\pi(|k|-k_0))\,e^{-\pi(|k|-k_0)}}{\pi(|k|-k_0)} \le 1,$$

we estimate[11]

$$\begin{aligned}\left|G^{-1}(k)\right| &= \frac{2\left(k^2+1\right)}{\left|k^2-k_0^2\right|\left[1+\coth\left(\pi\left(|k|-k_0\right)\right)\right]} = \frac{2\left(k^2+1\right)}{\left|k^2-k_0^2\right|}\sinh\left(\pi\left(|k|-k_0\right)\right)e^{-\pi(|k|-k_0)} \\ &\le 1-e^{-2\pi(|k|-k_0)}+\frac{2\pi\left(k_0^2+1\right)}{|k|+k_0} \le 1+2\pi\left(k_0+\frac{1}{k_0}\right) = \mathcal{O}\left(\log\lambda+\frac{1}{\log\lambda}\right).\end{aligned}$$

Therefore, using Cauchy-Schwarz inequality, isometry property of Fourier transform and the estimate (2.74), we have

$$\tilde{\varepsilon}_0 := \left|\int_{\mathbb{R}}\frac{\hat{\mathcal{R}}_0(t)/X_-(t)}{t-i}dt\right| \le \pi\left\|X_-^{-1}\right\|_{L^\infty(\mathbb{R})}\left\|\hat{\mathcal{R}}_0\right\|_{L^2(\mathbb{R})} = \pi\left\|G^{-1/2}\right\|_{L^\infty(\mathbb{R})}\|\mathcal{R}\|_{L^2(\mathbb{R}_+)} = \mathcal{O}\left(\frac{\left(\log^2\lambda+1\right)^{1/2}\log^{3/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\beta^{3/2}\right). \tag{2.97}$$

From (2.65), using definitions (2.75), we can express

$$\begin{aligned}\hat{\mathcal{M}}_0(k) &= \widehat{\chi_{\mathbb{R}_+}m''}(k)+4\pi^2k_0^2\widehat{\chi_{\mathbb{R}_+}m}-\mathcal{F}\chi_{\mathbb{R}_+}\mathcal{F}^{-1}\left[\hat{\mathcal{K}}\,\widehat{\chi_{\mathbb{R}_+}m}\right](k) \\ &= \varphi\left(\frac{1}{\beta}\right)\mathcal{A}(k)+\varphi'\left(\frac{1}{\beta}\right)\mathcal{B}(k),\end{aligned}$$

where

$$\mathcal{A}(k) := \frac{1+4\pi^2k_0^2}{(1-2\pi ik)^2}-\frac{1-4\pi^2k_0^2}{1-2\pi ik}-2P_+\left[\frac{1-\pi i\cdot}{(1-2\pi i\cdot)^2}\hat{\mathcal{K}}(\cdot)\right](k), \tag{2.98}$$

$$\mathcal{B}(k) := \frac{1+4\pi^2k_0^2}{(1-2\pi ik)^2}-\frac{2}{1-2\pi ik}-P_+\left[\frac{1}{(1-2\pi i\cdot)^2}\hat{\mathcal{K}}(\cdot)\right](k). \tag{2.99}$$

Therefore, (2.96) results in (2.85) with[12]

$$\varepsilon_0 := \frac{\tilde{\varepsilon}_0}{\int_{\mathbb{R}}\mathcal{B}(k)\left[(k-i)X_-(k)\right]^{-1}dk} = \mathcal{O}\left(\frac{(1+\log\lambda)\log^{3/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\beta^{3/2}\right), \tag{2.100}$$

and $\kappa\in\mathbb{R}$ being a λ-dependent constant defined as

$$\kappa := -\frac{P_+\left[\mathcal{A}/X_-\right](i)}{P_+\left[\mathcal{B}/X_-\right](i)} = -\frac{\int_{\mathbb{R}}\mathcal{A}(k)\left[(k-i)X_-(k)\right]^{-1}dk}{\int_{\mathbb{R}}\mathcal{B}(k)\left[(k-i)X_-(k)\right]^{-1}dk}.$$

We are going to show that this can be brought to an expression of a much simple form, namely, (2.80).

We start working out the denominator integral aiming to show that

$$\int_{\mathbb{R}}\mathcal{B}(k)\left[(k-i)X_-(k)\right]^{-1}dk = -2\pi i. \tag{2.101}$$

[11] It is at this point when we first see, due to the appearance of k_0 in denominators, worsening of the asymptotic estimates for those eigenvalues close to 1. To see that inequalities here cannot be qualitatively improved, observe that $G^{-1}(0)=\mathcal{O}(1/k_0)$ for small k_0.

[12] The denominator here is $\mathcal{O}(1)$, see (2.101) in further computations of κ.

First of all, we employ residue calculus to compute

$$\int_{\mathbb{R}} \frac{1}{k+\frac{i}{2\pi}} \frac{dk}{(k-i)\, X_-(k)} = \frac{4\pi^2}{1+2\pi} \frac{1}{X_-\left(-\frac{i}{2\pi}\right)}, \tag{2.102}$$

$$\int_{\mathbb{R}} \frac{1}{\left(k+\frac{i}{2\pi}\right)^2} \frac{dk}{(k-i)\, X_-(k)} = -2\pi i \left[\frac{1}{(k-i)\, X_-(k)}\right]' \left(-\frac{i}{2\pi}\right), \tag{2.103}$$

yielding

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{B}(k) \left[(k-i)\, X_-(k)\right]^{-1} dk \quad &= \quad \frac{2\pi i \left(1+4\pi^2 k_0^2\right)}{4\pi^2} \left[\frac{1}{(k-i)\, X_-(k)}\right]' \left(-\frac{i}{2\pi}\right) - \frac{4\pi i}{1+2\pi} \frac{1}{X_-\left(-\frac{i}{2\pi}\right)} \\ &\quad - \int_{\mathbb{R}} P_+ \left[\frac{1}{(1-2\pi i \cdot)^2} \hat{\mathcal{K}}(\cdot)\right](k) \frac{dk}{(k-i)\, X_-(k)}. \end{aligned} \tag{2.104}$$

Next, we observe that since

$$\frac{1}{(k-i)\, X_-(k)} = \overline{\frac{1}{(k+i)\, X_+(k)}},$$

in the P_+ term, the projection operator can be removed as the corresponding integral is an inner product taken against an H_+ function.

It, therefore, remains to consider

$$\int_{\mathbb{R}} \frac{\hat{\mathcal{K}}(k)}{(1-2\pi i k)^2} \frac{dk}{(k-i)\, X_-(k)} = \int_{\mathbb{R}} \frac{1}{\left(k+\frac{i}{2\pi}\right)^2} \frac{k^2-k_0^2-\left(k^2+1\right) G(k)}{(k-i)\, X_-(k)} dk.$$

Recalling factorization (2.90), non-vanishing of $X_-(k)$ and the fact that $X_\pm(k) \to 1$ as $|k| \to \infty$, we reorganize the terms in order to have an integrable decay of integrands at infinity and employ residue calculus for computation of each integral

$$\begin{aligned} \int_{\mathbb{R}} \frac{\hat{\mathcal{K}}(k)}{(1-2\pi i k)^2} \frac{dk}{(k-i)\, X_-(k)} \quad &= \quad \int_{\mathbb{R}} \frac{k^2\,(1-X_+(k))}{\left(k+\frac{i}{2\pi}\right)^2 (k-i)} dk + \int_{\mathbb{R}} \frac{k^2\,(1/X_-(k)-1)}{\left(k+\frac{i}{2\pi}\right)^2 (k-i)} dk \\ &\quad - \int_{\mathbb{R}} \frac{X_+(k)}{\left(k+\frac{i}{2\pi}\right)^2 (k-i)} dk - k_0^2 \int_{\mathbb{R}} \frac{1/X_-(k)}{\left(k+\frac{i}{2\pi}\right)^2 (k-i)} dk \\ &= \quad 2\pi i \Bigg(k_0^2 \left[\frac{1}{(k-i)\, X_-(k)}\right]' \left(-\frac{i}{2\pi}\right) - \left[\frac{k^2}{(k-i)\, X_-(k)}\right]' \left(-\frac{i}{2\pi}\right) \\ &\quad + \frac{1}{\left(1+\frac{1}{2\pi}\right)^2} - \frac{1}{(1+2\pi)^2} + \frac{2}{1+2\pi} \Bigg) \end{aligned}$$

Using this in (2.104), after cancellation of the X_- terms, we obtain (2.101).

Following the same pattern, we compute

$$\int_{\mathbb{R}} \mathcal{A}(k) \left[(k-i)\, X_-(k)\right]^{-1} dk \quad = \quad 2\pi i \left(\frac{1+4\pi^2 k_0^2}{4\pi^2} \left[\frac{1}{(k-i)\, X_-(k)}\right]' \left(-\frac{i}{2\pi}\right) - \frac{1-4\pi^2 k_0^2}{(1+2\pi)\, X_-\left(-\frac{i}{2\pi}\right)} - J \right),$$

where

$$\begin{aligned}
J &:= \int_{\mathbb{R}} \frac{k+\frac{i}{\pi}}{\left(k+\frac{i}{2\pi}\right)^2} \frac{-k^2+k_0^2+\left(k^2+1\right)X_+(k)\,X_-(k)}{(k-i)\,X_-(k)}dk \\
&= \int_{\mathbb{R}} \frac{\frac{k}{\pi}\left(\frac{1}{4\pi}-1\right)+i\left(k^2-\frac{1}{4\pi^2}\right)}{\left(k+\frac{i}{2\pi}\right)^2(k-i)}\left(X_+(k)-1\right)dk + \int_{\mathbb{R}} \frac{\frac{k}{\pi}\left(\frac{1}{4\pi}-1\right)+i\left(k^2-\frac{1}{4\pi^2}\right)}{\left(k+\frac{i}{2\pi}\right)^2(k-i)}\left(1-\frac{1}{X_-(k)}\right)dk \\
&\quad + \int_{\mathbb{R}} \frac{\left(k+\frac{i}{\pi}\right)X_+(k)}{\left(k+\frac{i}{2\pi}\right)^2(k-i)}dk + k_0^2\int_{\mathbb{R}} \frac{\left(k+\frac{i}{\pi}\right)1/X_-(k)}{\left(k+\frac{i}{2\pi}\right)^2(k-i)}dk + \int_{\mathbb{R}}\left[X_+(k)-\frac{1}{X_-(k)}\right]dk.
\end{aligned}$$

As before, all the terms can be computed using residue calculus while the last term deserves a special attention. We base its computation on the asymptotical behavior

$$X_{\pm}(k) = 1 \mp \frac{\omega}{i\pi k} + \mathcal{O}\left(\frac{1}{k^3}\right), \tag{2.105}$$

where

$$\omega := \int_0^\infty \log G(s)\,ds, \tag{2.106}$$

which is a consequence of the asymptotics of the Hilbert transform

$$⨍_{\mathbb{R}} \frac{\log G(s)}{s-k}ds = -\frac{2\omega}{k} + \mathcal{O}\left(\frac{1}{k^3}\right). \tag{2.107}$$

The latter can be obtained taking into account that

$$\log G(k) = -\frac{1}{k^2+1}\left(k_0^2+1\right) + \mathcal{O}\left(\frac{1}{k^4}\right),$$

and

$$\frac{1}{\pi}⨍_{\mathbb{R}} \frac{1}{\tau^2+1}\frac{d\tau}{\tau-k} = -\frac{k}{k^2+1}.$$

Namely, writing

$$\frac{1}{\pi}⨍_{\mathbb{R}} \frac{\log G(\tau)}{\tau-k}d\tau = \frac{1}{\pi}⨍_{\mathbb{R}}\left[\log G(\tau)+\frac{k_0^2+1}{\tau^2+1}\right]\frac{d\tau}{\tau-k} + \frac{k\left(k_0^2+1\right)}{k^2+1},$$

we are in position to apply Lemma 2.4.1 to get the asymptotics of the integral term

$$\frac{1}{\pi}⨍_{\mathbb{R}}\left[\log G(\tau)+\frac{k_0^2+1}{\tau^2+1}\right]\frac{d\tau}{\tau-k} = -\frac{2}{\pi k}\int_0^\infty \log G(s)\,ds - \frac{k_0^2+1}{k} + \mathcal{O}\left(\frac{1}{k^3}\right),$$

leading to (2.107) and hence to (2.105).

Now we are ready to come back to evaluation

$$\int_{\mathbb{R}}\left[X_+(k)-\frac{1}{X_-(k)}\right]dk = ⨍_{\mathbb{R}}\left[X_+(k)-1+\frac{\omega}{i\pi k}\right]dk + ⨍_{\mathbb{R}}\left[1-\frac{1}{X_-(k)}-\frac{\omega}{i\pi k}\right]dk = 2\omega,$$

where we indented contours in both integrals at $k=0$ and used Cauchy theorem to deduce vanishing of the resulting integrals, whereas integration over the half-circle indention in each integral is responsible for the ω contribution.

Finally,

$$
\begin{aligned}
J &= \frac{2\pi\left(1+\frac{1}{\pi}\right)}{\left(1+\frac{1}{2\pi}\right)^2} - \frac{1}{(1+2\pi)\,X_-\left(-\frac{i}{2\pi}\right)} + \frac{1}{4\pi^2}\left[\frac{1}{(k-i)\,X_-(k)}\right]'\left(-\frac{i}{2\pi}\right) \\
&\quad + \frac{2\pi k_0^2}{\left(1+\frac{1}{2\pi}\right)X_-\left(-\frac{i}{2\pi}\right)} + k_0^2\left[\frac{1}{(k-i)\,X_-(k)}\right]'\left(-\frac{i}{2\pi}\right) + \frac{1}{2\pi\left(1+\frac{1}{2\pi}\right)^2} + 2\omega,
\end{aligned}
$$

and therefore we conclude

$$\int_{\mathbb{R}} \mathcal{A}(k)\left[(k-i)\,X_-(k)\right]^{-1} dk = -4\pi i\,(\pi+\omega)\,,$$

and

$$\kappa = -2\,(\pi+\omega) = -2\pi - 2\int_0^\infty \log G(s)\,ds.$$

Taking inverse Fourier transform of (2.95) and absorbing the factor $1/X_+ \in H_+$ into P_+ operator, we obtain the solution

$$
\begin{aligned}
\varphi_0(x) &:= \int_{\mathbb{R}} e^{-2\pi ikx}\frac{P_+\left[\hat{\mathcal{M}}_0/X_-\right](k)}{4\pi^2\left(k^2+1\right)X_+(k)}dk + \mathcal{E}_0^{(1)}(x) = \int_{\mathbb{R}} e^{-2\pi ikx}\frac{P_+\left[\hat{\mathcal{M}}_0/G\right](k)}{4\pi^2\left(k^2+1\right)}dk + \mathcal{E}_0^{(1)}(x) \\
&= \varphi\left(\frac{1}{\beta}\right)\int_{\mathbb{R}} e^{-2\pi ikx}\frac{P_+\left[\mathcal{C}/G\right](k)}{4\pi^2\left(k^2+1\right)}dk + \mathcal{E}_0(x)
\end{aligned}
\tag{2.108}
$$

with the error term

$$\mathcal{E}_0(x) := \mathcal{E}_0^{(1)}(x) + \mathcal{E}_0^{(2)}(x)\,, \tag{2.109}$$

$$\mathcal{E}_0^{(1)}(x) := \int_{\mathbb{R}} e^{-2\pi ikx}\frac{P_+\left[\hat{\mathcal{R}}_0/G\right](k)}{4\pi^2\left(k^2+1\right)}dk, \tag{2.110}$$

$$\mathcal{E}_0^{(2)}(x) := \varepsilon_0\int_{\mathbb{R}} e^{-2\pi ikx}\frac{P_+\left[\mathcal{B}/G\right](k)}{4\pi^2\left(k^2+1\right)}dk, \tag{2.111}$$

and $\mathcal{C}(k) := \mathcal{A}(k) + \kappa\mathcal{B}(k)$ is as in (2.81).

The error term can be estimated using the same ingredients as for the bound (2.97) along with L^2-boundedness of the projection operator P_+

$$
\begin{aligned}
\left\|\mathcal{E}_0^{(1)}\right\|_{W^{2,2}(\mathbb{R}_+)} &= \left[\int_{\mathbb{R}}\left(\frac{\left(4\pi^2k^2+1\right)P_+\left[\hat{\mathcal{R}}_0/G\right](k)}{4\pi^2\left(k^2+1\right)}\right)^2 dk\right]^{1/2} \\
&\le \left\|G^{-1}\right\|_{L^\infty(\mathbb{R})}\left\|P_+\left[\hat{\mathcal{R}}_0\right]\right\|_{L^2(\mathbb{R})} \le \left\|G^{-1}\right\|_{L^\infty(\mathbb{R})}\left\|\mathcal{R}\right\|_{L^2(\mathbb{R}_+)} \\
&= \mathcal{O}\left(\frac{\left(1+\log^2\lambda\right)\log\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\beta^{3/2}\right),
\end{aligned}
$$

and similarly,

$$\left\|\mathcal{E}_0^{(2)}\right\|_{W^{2,2}(\mathbb{R}_+)} \leq \varepsilon_0 \left\|G^{-1}\right\|_{L^\infty(\mathbb{R})} \|\mathcal{B}\|_{L^2(\mathbb{R})} \leq \varepsilon_0 \left\|G^{-1}\right\|_{L^\infty(\mathbb{R})} \left(1 + k_0^2 + \left\|\hat{\mathcal{K}}\right\|_{L^2(\mathbb{R})}\right),$$

where the $\hat{\mathcal{K}}$ term can be estimated by means of

$$\begin{aligned} \left\|\hat{\mathcal{K}}\right\|_{L^2(\mathbb{R})}^2 &= 2\int_0^\infty \hat{\mathcal{K}}^2(k)\,dk = 16\pi^3 \int_0^\infty \frac{(k-k_0)^2}{\sinh^2(\pi(k-k_0))}(k+k_0)^2 e^{-2\pi(k-k_0)}dk \\ &\leq 16\pi e^{4\pi k_0}\int_0^\infty k^2 e^{-2\pi k}dk = 4e^{2\pi k_0}\left(2k_0^2 + \frac{2k_0}{\pi} + \frac{1}{\pi^2}\right) = \mathcal{O}\left(\frac{1+\log^2\lambda}{\lambda}\right) \end{aligned} \tag{2.112}$$

due to the inequality

$$\frac{(k-k_0)^2}{\sinh^2(\pi(k-k_0))} \leq \frac{1}{\pi^2}.$$

Therefore,

$$\begin{aligned} \left\|\mathcal{E}_0^{(2)}\right\|_{W^{2,2}(\mathbb{R}_+)} &= \mathcal{O}\left(\frac{(1+\log\lambda)\log^{3/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\left(\log\lambda + \frac{1}{\log\lambda}\right)\left(1+\log^2\lambda + \frac{1+\log\lambda}{\lambda^{1/2}}\right)\beta^{3/2}\right) \\ &= \mathcal{O}\left(\frac{(1+\log\lambda)\left(1+\log^2\lambda\right)\log^{1/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\left(1+\log^2\lambda + \frac{1+\log\lambda}{\lambda^{1/2}}\right)\beta^{3/2}\right) \\ &= \mathcal{O}\left(\frac{\log^{1/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\left(1+\log^5\lambda + \frac{1+\log^4\lambda}{\lambda^{1/2}}\right)\beta^{3/2}\right), \end{aligned}$$

and hence

$$\|\mathcal{E}_0\|_{W^{2,2}(\mathbb{R}_+)} = \mathcal{O}\left(\frac{\log^{1/2}\lambda}{\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\left(1+\log^5\lambda + \frac{1+\log^4\lambda}{\lambda^{1/2}} + \log^{1/2}\lambda\left(1+\log^2\lambda\right)\right)\beta^{3/2}\right) = \mathcal{O}\left(C_\lambda\beta^{3/2}\right).$$

□

Remark 2.3.1. *By Sobolev embedding (Morrey's inequality [12, Sect. 5.6 Thm 4]), we have, for some constant $C>0$ independent of λ and β,*

$$\|\mathcal{E}_0\|_{L^\infty(\mathbb{R}_+)}, \|\mathcal{E}_0'\|_{L^\infty(\mathbb{R}_+)} \leq C\|\mathcal{E}_0\|_{W^{2,2}(\mathbb{R}_+)} = \mathcal{O}\left(C_\lambda\beta^{3/2}\right). \tag{2.113}$$

Exactly by the same reasoning applied to the first term in (2.108), we deduce that the approximate solution $(\varphi_0 - \mathcal{E}_0)$ belongs to $W^{2,2}(\mathbb{R}_+)$ and hence also to $C^1(\mathbb{R}_+)$ making pointwise values of $(\varphi_0-\mathcal{E}_0)$, $(\varphi_0-\mathcal{E}_0)'$ well-defined.

Remark 2.3.2. *Even though we have pointwise control of the approximation of the solution along with its derivative, evaluation of $(\varphi_0 - \mathcal{E}_0)(0)$, $(\varphi_0 - \mathcal{E}_0)'(0)$ from the obtained expression (2.84) using that $\varphi_0(0) = \varphi_0'(0) = 0$ does not bring any new information. Indeed, vanishing of the exponential factor at $x = 0$ allows us to perform integration by employing residue calculus, and thus we can see that each of the equalities $\varphi_0(0) = 0$ and $\varphi_0'(0) = 0$ is equivalent to the boundary condition (2.85) that has already been found.*

Setting, for $x > 1/\beta$,

$$\mathcal{E}(x) := \mathcal{E}_0\left(x - \frac{1}{\beta}\right) + \varepsilon_0 e^{-x+1/\beta}\left(x - \frac{1}{\beta}\right), \tag{2.114}$$

the result of Proposition 2.3.1 can be rephrased as

Proposition 2.3.2. *For $\beta \ll 1$, analytic continuation of the solution (2.38) normalized as $\|\varphi\|_{L^2(B)} = 1$ is given, for $x > 1/\beta$, by*

$$\varphi(x) \quad = \quad \varphi\left(\frac{1}{\beta}\right)\left[e^{-x+1/\beta}\left(1 + (1+\kappa)(x - 1/\beta)\right) + \int_{\mathbb{R}} e^{-2\pi i k(x-1/\beta)} \frac{P_+\left[\mathcal{C}/G\right](k)}{4\pi^2(k^2+1)} dk\right] + \mathcal{E}(x), \tag{2.115}$$

where $\|\mathcal{E}\|_{W^{2,2}(1/\beta,\infty)} = \mathcal{O}\left(C_\lambda \beta^{3/2}\right)$.

Now it remains to recover the solution inside the interval B.

It is tempting to employ Lemma 2.3.1 providing a direct analytic continuation from $\mathbb{R}\backslash B$, however, the oscillatory behaviour of the kernel R_0 at infinity and lack of $L^1(1/\beta, \infty)$ estimates for the constructed approximant prevents us from bounding the error term. Instead, we will resort to (2.60) and construct solution by inversion of a simple differential operator.

To this effect, let us introduce

$$N^{\pm}(x) := \int_{1/\beta}^{\infty} \left[\mathcal{K}(x-t) \pm \mathcal{K}(x+t)\right] \varphi(t)\, dt, \tag{2.116}$$

where the upper sign on the left corresponds to even parity of φ and the lower one to odd φ.

Due to (2.115), this quantity is known up to the constant $\varphi\left(\frac{1}{\beta}\right)$.

Expression (2.116) is precisely the left-hand side of (2.60) with the sign chosen according to the parity of a solution to be constructed.

Now, when $x \in B$, instead of an integro-differential equation, (2.60) becomes an elementary ODE which can be solved by the method of variation of parameters

$$\varphi(x) = C_1 \cos(2\pi k_0 x) + C_2 \sin(2\pi k_0 x) + \frac{\sin(2\pi k_0 x)}{2\pi k_0} \int_0^x N^{\pm}(t) \cos(2\pi k_0 t)\, dt - \frac{\cos(2\pi k_0 x)}{2\pi k_0} \int_0^x N^{\pm}(t) \sin(2\pi k_0 t)\, dt,$$

Namely, even and odd solutions satisfy, respectively, for $x \in B$,

$$\begin{aligned} \varphi_{even}(x) &= C_1 \cos(2\pi k_0 x) + \frac{1}{2\pi k_0}\int_0^x N^+(t)\sin(2\pi k_0(x-t))\,dt, \\ &= C_1 \cos(2\pi k_0 x) + \varphi\left(\frac{1}{\beta}\right)\int_0^x N_0^+(t)\sin(2\pi k_0(x-t))\,dt + \int_0^x \mathcal{E}^+(t)\sin(2\pi k_0(x-t))\,dt, \end{aligned} \tag{2.117}$$

$$\begin{aligned} \varphi_{odd}(x) &= C_2 \sin(2\pi k_0 x) + \frac{1}{2\pi k_0}\int_0^x N^-(t)\sin(2\pi k_0(x-t))\,dt, \\ &= C_2 \sin(2\pi k_0 x) + \varphi\left(\frac{1}{\beta}\right)\int_0^x N_0^-(t)\sin(2\pi k_0(x-t))\,dt + \int_0^x \mathcal{E}^-(t)\sin(2\pi k_0(x-t))\,dt, \end{aligned} \tag{2.118}$$

where

$$\begin{aligned} N_0^{\pm}(x) &:= \frac{1}{2\pi k_0}\int_0^\infty \left(\mathcal{K}\left(x-t-\frac{1}{\beta}\right) \pm \mathcal{K}\left(x+t+\frac{1}{\beta}\right)\right)\Bigg[\left(1+(1+\kappa)\,te^{-t}\right) \\ &\quad + \int_{\mathbb{R}} e^{-2\pi ikt}\frac{P_+[\mathcal{C}/G](k)}{4\pi^2(k^2+1)}dk\Bigg]dt, \end{aligned}$$

$$\mathcal{E}^{\pm}(x) := \frac{1}{2\pi k_0}\int_0^\infty \left(\mathcal{K}\left(x-t-\frac{1}{\beta}\right) \pm \mathcal{K}\left(x+t+\frac{1}{\beta}\right)\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt.$$

We are now aiming to derive estimates for the $\mathcal{E}^{\pm}$ error terms in (2.117)-(2.118) for $x \in B$. By parity reasons, it suffices to do so only for $x \in \left[0, \frac{1}{\beta}\right]$. And, due to the full analogy between even and odd cases, we consider only the estimates for (2.117).

We first focus on the terms involving $\mathcal{K}\left(x+t+\frac{1}{\beta}\right)$, and using the asymptotic behavior (2.72), we estimate, for $\tau \in \left(0, \frac{1}{\beta}\right)$,

$$\int_0^\infty \mathcal{K}\left(\tau+t+\frac{1}{\beta}\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt \lesssim \|\mathcal{E}\|_{L^\infty\left(\frac{1}{\beta},\infty\right)}\frac{\pi k_0^2}{\sinh^2(\pi k_0)}\frac{1}{\tau+1/\beta}, \tag{2.119}$$

and hence, after integration in τ, using that $\|\mathcal{E}\|_{L^\infty\left(\frac{1}{\beta},\infty\right)} = \mathcal{O}\left(C_\lambda \beta^{3/2}\right)$ (recall (2.113)), we get

$$\frac{1}{2\pi k_0}\left|\int_0^x\int_0^\infty \mathcal{K}\left(\tau+t+\frac{1}{\beta}\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt\sin(2\pi k_0(x-\tau))\,d\tau\right| = \mathcal{O}\left(\frac{C_\lambda \log\lambda}{\sinh^2\left(\frac{1}{2}\log\lambda\right)}\beta^{3/2}\log\beta\right).$$

To deal with the part involving $\mathcal{K}\left(x-t-\frac{1}{\beta}\right)$, we split the interval as $\left[0, \frac{1}{\beta}\right] = \left[0, \frac{1}{\beta}-\frac{1}{\beta^\alpha}\right] \cup \left[\frac{1}{\beta}-\frac{1}{\beta^\alpha}, \frac{1}{\beta}\right]$, for some $\alpha > 0$, and consider first the situation $x \in \left[0, \frac{1}{\beta}-\frac{1}{\beta^\alpha}\right]$. Using the gap between the ranges of the integration variable t and x, we can again take advantage of the asymptotic decay (2.72) to estimate as before

$$\frac{1}{2\pi k_0}\left|\int_0^x\int_0^\infty \mathcal{K}\left(\tau-t-\frac{1}{\beta}\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt\sin(2\pi k_0(x-\tau))\,d\tau\right| = \mathcal{O}\left(\frac{C_\lambda \log\lambda}{\sinh^2\left(\frac{1}{2}\log\lambda\right)}\beta^{3/2}\alpha\log\beta\right).$$

Now, for $x \in \left[\frac{1}{\beta}-\frac{1}{\beta^\alpha}, \frac{1}{\beta}\right]$, we split the integration range and rely on the above result for $x \in \left[0, \frac{1}{\beta}-\frac{1}{\beta^\alpha}\right]$ in the first integral and the smallness of the interval in the second one (we also employ isometry of the Fourier transform and the estimate (2.112)), namely,

$$\frac{1}{2\pi k_0}\left|\int_0^{\frac{1}{\beta}-\frac{1}{\beta^\alpha}}\int_0^\infty \mathcal{K}\left(\tau-t-\frac{1}{\beta}\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt\sin\left(2\pi k_0\left(x-\tau\right)\right)d\tau\right| = \mathcal{O}\left(\frac{C_\lambda\log\lambda}{\sinh^2\left(\frac{1}{2}\log\lambda\right)}\beta^{3/2}\alpha\log\beta\right).$$

$$\begin{aligned}\frac{1}{2\pi k_0}\left|\int_{\frac{1}{\beta}-\frac{1}{\beta^\alpha}}^{x}\int_0^\infty \mathcal{K}\left(\tau-t-\frac{1}{\beta}\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt\sin\left(2\pi k_0\left(x-\tau\right)\right)d\tau\right| &\le \frac{1}{2\pi k_0\beta^\alpha}\left\|\mathcal{K}\right\|_{L^2(\mathbb{R})}\left\|\mathcal{E}\right\|_{L^2\left(\frac{1}{\beta},\infty\right)}\\ &= \mathcal{O}\left(\frac{1+\log\lambda}{\lambda^{1/2}\log\lambda}C_\lambda\beta^{3/2-\alpha}\right),\end{aligned}$$

leading to

$$\frac{1}{2\pi k_0}\left|\left(\int_0^{\frac{1}{\beta}-\frac{1}{\beta^\alpha}}+\int_{\frac{1}{\beta}-\frac{1}{\beta^\alpha}}^{x}\right)\int_0^\infty \mathcal{K}\left(\tau-t-\frac{1}{\beta}\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt\sin\left(2\pi k_0\left(x-\tau\right)\right)d\tau\right| = \mathcal{O}\left(C_\lambda^{(1)}\beta^{3/2-\alpha}+C_\lambda^{(2)}\beta^{3/2}\log\beta\right),$$

where, recalling (2.82),

$$C_\lambda^{(1)} := \frac{1+\log\lambda}{\lambda^{1/2}\log\lambda}C_\lambda = \frac{1+\log\lambda}{\lambda^{3/2}\log^{1/2}\lambda\sinh^2\left(\frac{1}{2}\log\lambda\right)}\left(1+\log^5\lambda+\frac{1+\log^4\lambda}{\lambda^{1/2}}\right), \tag{2.120}$$

$$C_\lambda^{(2)} := \frac{\log\lambda}{\sinh^2\left(\frac{1}{2}\log\lambda\right)}C_\lambda = \frac{\log^{3/2}\lambda}{\lambda\sinh^4\left(\frac{1}{2}\log\lambda\right)}\left(1+\log^5\lambda+\frac{1+\log^4\lambda}{\lambda^{1/2}}\right). \tag{2.121}$$

Therefore, for $x\in\left[0,\frac{1}{\beta}\right]$,

$$\frac{1}{2\pi k_0}\left|\int_0^x\int_0^\infty \mathcal{K}\left(\tau-t-\frac{1}{\beta}\right)\mathcal{E}\left(t+\frac{1}{\beta}\right)dt\sin\left(2\pi k_0\left(x-\tau\right)\right)d\tau\right| = \mathcal{O}\left(C_\lambda^{(1)}\beta^{3/2-\alpha}+C_\lambda^{(2)}\beta^{3/2}\log\beta\right),$$

and combining this with (2.119), we deduce the same error order estimate for the whole error term in (2.117), for $x\in B$,

$$\left|\int_0^x\mathcal{E}^+\left(t\right)\sin\left(2\pi k_0\left(x-t\right)\right)dt\right| = \mathcal{O}\left(C_\lambda^{(1)}\beta^{3/2-\alpha}+C_\lambda^{(2)}\beta^{3/2}\log\beta\right),$$

Similarly, for the derivative,

$$\begin{aligned}\left|\frac{d}{dx}\int_0^x\mathcal{E}^+\left(t\right)\sin\left(2\pi k_0\left(x-t\right)\right)dt\right| &= 2\pi k_0\left|\int_0^x\mathcal{E}^+\left(t\right)\cos\left(2\pi k_0\left(x-t\right)\right)dt\right|\\ &= \mathcal{O}\left(C_\lambda^{(1)}\log\lambda\beta^{3/2-\alpha}+C_\lambda^{(2)}\log\lambda\beta^{3/2}\log\beta\right).\end{aligned}$$

We, therefore, have

$$\begin{aligned}\varphi_{even}\left(x\right) &= C_1\cos\left(2\pi k_0x\right)+\varphi\left(\frac{1}{\beta}\right)\int_0^x N_0^+\left(t\right)\sin\left(2\pi k_0\left(x-t\right)\right)dt\\ &\quad+\mathcal{O}\left(C_\lambda^{(1)}\beta^{3/2-\alpha}+C_\lambda^{(2)}\beta^{3/2}\log\beta\right),\end{aligned} \tag{2.122}$$

$$\begin{aligned}\varphi_{even}'(x) &= -2\pi k_0 C_1 \sin(2\pi k_0 x) + 2\pi k_0 \varphi\left(\frac{1}{\beta}\right)\int_0^x N_0^+(t)\cos(2\pi k_0(x-t))\,dt \\ &+\mathcal{O}\left(C_\lambda^{(1)}\log\lambda\beta^{3/2-\alpha} + C_\lambda^{(2)}\log\lambda\beta^{3/2}\log\beta\right).\end{aligned} \tag{2.123}$$

Evaluation of (2.122) at $x = 1/\beta$ yields

$$C_1 = \frac{\varphi(1/\beta)}{\cos(2\pi k_0/\beta)}\left[1 - \int_0^{\frac{1}{\beta}} N_0^+(t)\sin\left(2\pi k_0\left(\frac{1}{\beta}-t\right)\right)dt\right] + \mathcal{O}\left(C_\lambda^{(1)}\beta^{3/2-\alpha} + C_\lambda^{(2)}\beta^{3/2}\log\beta\right), \tag{2.124}$$

and, using this and (2.85) in (2.123), we arrive at the approximate characteristic equation to be solved for $k_0 = -\dfrac{\log\lambda}{2\pi}$ and hence eigenvalues corresponding to even eigenfunctions

$$\begin{aligned}\frac{\kappa}{2\pi k_0} &= -\tan(2\pi k_0/\beta)\left[1 - \int_0^{\frac{1}{\beta}} N_0^+(t)\sin\left(2\pi k_0\left(\frac{1}{\beta}-t\right)\right)dt\right] + \int_0^{\frac{1}{\beta}} N_0^+(t)\cos\left(2\pi k_0\left(\frac{1}{\beta}-t\right)\right)dt, \\ &+\mathcal{O}\left((1+\log\lambda)\left(C_\lambda^{(1)}\beta^{3/2-\alpha} + C_\lambda^{(2)}\beta^{3/2}\log\beta\right) + C_\lambda^{(0)}\beta^{3/2}\right),\end{aligned}$$

and, after simplification,

$$\begin{aligned}\frac{\kappa}{2\pi k_0}\cos(2\pi k_0/\beta) + \sin(2\pi k_0/\beta) &= \int_0^{\frac{1}{\beta}} N_0^+(t)\cos(2\pi k_0 t)\,dt \\ &+\mathcal{O}\left((1+\log\lambda)\left(C_\lambda^{(1)}\beta^{3/2-\alpha} + C_\lambda^{(2)}\beta^{3/2}\log\beta\right) + C_\lambda^{(0)}\beta^{3/2}\right).\end{aligned} \tag{2.125}$$

Similarly, from boundary values of

$$\begin{aligned}\varphi_{odd}(x) &= C_2\sin(2\pi k_0 x) + \varphi\left(\frac{1}{\beta}\right)\int_0^x N_0^-(t)\sin(2\pi k_0(x-t))\,dt \\ &+\mathcal{O}\left(C_\lambda^{(1)}\beta^{3/2-\alpha} + C_\lambda^{(2)}\beta^{3/2}\log\beta\right),\end{aligned} \tag{2.126}$$

$$\varphi_{odd}'(x) = 2\pi k_0 C_2\cos(2\pi k_0 x) + 2\pi k_0\varphi\left(\frac{1}{\beta}\right)\int_0^x N_0^-(t)\cos(2\pi k_0(x-t))\,dt \tag{2.127}$$

$$+\mathcal{O}\left(C_\lambda^{(1)}\log\lambda\beta^{3/2-\alpha} + C_\lambda^{(2)}\log\lambda\beta^{3/2}\log\beta\right). \tag{2.128}$$

at $x = 1/\beta$, we obtain

$$C_2 = \frac{\varphi(1/\beta)}{\sin(2\pi k_0/\beta)}\left[1 - \int_0^{\frac{1}{\beta}} N_0^-(t)\sin\left(2\pi k_0\left(\frac{1}{\beta}-t\right)\right)dt\right] + \mathcal{O}\left(C_\lambda^{(1)}\beta^{3/2-\alpha} + C_\lambda^{(2)}\beta^{3/2}\log\beta\right), \tag{2.129}$$

and consequently, we derive a characteristic equation whose roots will give rise to odd eigenfunctions

$$\begin{aligned}\frac{\kappa}{2\pi k_0} &= \cot(2\pi k_0/\beta)\left[1 - \int_0^{\frac{1}{\beta}} N_0^-(t)\sin\left(2\pi k_0\left(\frac{1}{\beta}-t\right)\right)dt\right] + \int_0^{\frac{1}{\beta}} N_0^-(t)\cos\left(2\pi k_0\left(\frac{1}{\beta}-t\right)\right)dt \\ &+\mathcal{O}\left((1+\log\lambda)\left(C_\lambda^{(1)}\beta^{3/2-\alpha} + C_\lambda^{(2)}\beta^{3/2}\log\beta\right) + C_\lambda^{(0)}\beta^{3/2}\right)\end{aligned}$$

or, in a simplified form,

$$\begin{aligned}\frac{\kappa}{2\pi k_0}\sin\left(2\pi k_0/\beta\right)-\cos\left(2\pi k_0/\beta\right) &= \int_0^{\frac{1}{\beta}} N_0^-\left(t\right)\sin\left(2\pi k_0 t\right)dt \\ &+\mathcal{O}\left(\left(1+\log\lambda\right)\left(C_\lambda^{(1)}\beta^{3/2-\alpha}+C_\lambda^{(2)}\beta^{3/2}\log\beta\right)+C_\lambda^{(0)}\beta^{3/2}\right),\end{aligned} \tag{2.130}$$

where $\alpha>0$ is a constant that can be chosen arbitrary small.

We have thus just proved the main result of the approximation procedure.

Theorem 2.3.1. *For $\beta \ll 1$, eigenvalues $\{\lambda_n\}_{n=1,3,5,\ldots}$ corresponding to even eigenfunctions are the roots of the approximate characteristic equation (2.125) while approximations to even eigenfunctions are given by (2.122) with the constant C_1 defined in (2.124). In the odd parity case, approximate characteristic equation for eigenvalues $\{\lambda_n\}_{n=2,4,6,\ldots}$ is (2.130), corresponding eigenfunctions are approximately furnished by (2.126) and constant C_2 is given in (2.129). In both cases, the multiplicative factor $\varphi\left(1/\beta\right)$ is found from the normalization condition $\|\varphi\|_{L^2(B)}=1$.*

Remark 2.3.3. *As we see from the error term estimates, the asymptoticness breaks down in the limiting situations $\lambda\searrow 0$ and $\lambda\nearrow 1$. Therefore, we expect asymptotical approximation of the solutions to be really good in the middle range of eigenvalues. Numerical results confirm this fact as we further show in Section 2.4.*

Another problem on the half-line and connection to Keldysh-Lavrentiev equation

For the sake of completeness, we would like to point out another interesting link between the interval problem and a formulation on the half-line.

Proposition 2.3.3. *The analytic continuation of the solution of (2.13) given by (2.38) satisfies*

$$P_1\left[\chi_{\mathbb{R}\setminus B}\,\varphi\right](x)=-\lambda\varphi\left(x\right)+\varphi^-\left(x+i\right)+\varphi^+\left(x-i\right)-\frac{1}{\lambda}\chi_B\left(x\right)\varphi\left(x\right),\quad x\in\mathbb{R}, \tag{2.131}$$

where

$$\varphi^-\left(x+i\right):=\varphi\left(x+i-i0^+\right)=\lim_{\epsilon\searrow 0}\varphi\left(x+i-i\epsilon\right),$$

$$\varphi^+\left(x-i\right):=\varphi\left(x-i+i0^+\right)=\lim_{\epsilon\searrow 0}\varphi\left(x-i+i\epsilon\right).$$

Proof. Let us rewrite (2.38) as

$$\frac{1}{2\pi i}\int_B\left(\frac{1}{t-z-i}-\frac{1}{t-z+i}\right)\varphi\left(t\right)dt=\lambda\varphi\left(z\right),$$

and apply Plemelj-Sokhotski formulae for $z=x+i+i0^-$ and $z=x-i+i0^+$ to obtain, respectively, for $x\in\mathbb{R}$,

$$\frac{1}{2}\chi_B\left(x\right)\varphi\left(x\right)+\frac{i}{2\pi}\left[\fint_B\frac{\varphi\left(t\right)}{t-x}dt-\int_B\frac{\varphi\left(t\right)}{t-x-2i}dt\right]=\lambda\varphi^-\left(x+i\right), \tag{2.132}$$

$$\frac{1}{2}\chi_B(x)\varphi(x) - \frac{i}{2\pi}\left[\fint_B \frac{\varphi(t)}{t-x}dt - \int_B \frac{\varphi(t)}{t-x+2i}dt\right] = \lambda\varphi^+(x-i). \tag{2.133}$$

Upon adding and subtracting both expressions, we arrive at

$$\chi_B(x)\varphi(x) + P_2[\chi_B\varphi](x) = \lambda\left[\varphi^-(x+i) + \varphi^+(x-i)\right], \tag{2.134}$$

$$\frac{i}{\pi}\fint_B \frac{\varphi(t)}{t-x}dt + iQ_2[\chi_B\varphi](x) = \lambda\left[\varphi^-(x+i) - \varphi^+(x-i)\right], \tag{2.135}$$

where

$$Q_h[\chi_B\varphi](x) := \frac{1}{\pi}\int_B \frac{(x-t)\varphi(t)}{(x-t)^2+h^2}dt, \quad h>0,$$

is the conjugate Poisson operator [16].

Now applying P_1 to (2.38), we make use of the semigroup property of the Poisson operator [16, Ch. 1]

$$P_1[\chi_B\varphi](x) = \lambda\varphi(x) \quad \Rightarrow \quad P_2[\chi_B\varphi](x) = \lambda P_1[\varphi](x), \quad x\in\mathbb{R}.$$

Employing

$$P_1[\chi_B\varphi](x) = P_1[\varphi](x) - P_1[(1-\chi_B)\varphi](x) = \frac{1}{\lambda}P_2[\chi_B\varphi](x) - P_1\left[\chi_{\mathbb{R}\setminus B}\varphi\right](x),$$

and eliminating $P_2[\chi_B\varphi]$ from (2.134), we arrive at (2.131). □

Denoting Hilbert transform operator [16]

$$\mathcal{H}[f](x) := \frac{1}{\pi}\fint_{\mathbb{R}} \frac{f(t)}{x-t}dt, \tag{2.136}$$

we apply it to (2.134) and subtract (2.135). Since $\mathcal{H}[P_2[f]] = Q_2[f]$, we deduce that

$$\frac{1}{\pi}\fint_{\mathbb{R}} \frac{\varphi^-(t+i)+\varphi^+(t-i)}{x-t}dt = -i\left[\varphi^-(x+i) - \varphi^+(x-i)\right] + \frac{2}{\lambda}\mathcal{H}[\chi_B\varphi](x), \quad x\in\mathbb{R}.$$

Now, expressing the integrand on the left by means of (2.131) and the first term in the right-hand side from (2.135), we eventually arrive at

$$\frac{1}{\pi}\fint_B \frac{\varphi(t)}{t-x}dt + \frac{1}{\lambda^2}\int_B \frac{(x-t)\varphi(t)}{(x-t)^2+4}dt = \left(\mathcal{H} + \frac{1}{\lambda}Q_1\right)\left[\chi_{\mathbb{R}\setminus B}\varphi\right](x), \quad x\in B, \tag{2.137}$$

an equation whose particular instance is known as the limiting case of Keldysh-Lavrentiev equation [61] arising in hydrodynamical modelling of underwater wing motion [54]. This equation falls into a family of integral equations of the first kind on an interval (namely, convolution type with the Cauchy kernel plus an L^2 kernel) studied by Novokshenov. In [49], the solution has been given in terms of auxiliary matrix factors whose existence is proven in [19], yet constuctively unavailable in general. Asymptotical solution of (2.137) can be obtained following the work [48] once the rescaling of the argument of φ by $1/\beta$ is performed leading to the required appearance of a small parameter $4\beta^2$ in place of 4 in the denominator of the second integral term.

The described procedure provides an alternative possibility of constructing eigenfunctions from the solution of the half-line problem. However, it appears much more cumbersome than what we have done in proving Theorem 2.3.1.

Revisiting the previous result on pointwise conditions

We now look back at the result of Corollary 2.2.1 which can be simplified in the present approximation set-up $\beta \ll 1$.

Indeed, let us rewrite (2.25) as

$$\fint_0^\infty \hat{f}_0(it)\sin(2\pi ht)\,e^{-2\pi at}\left[\frac{e^{2\pi a\tau_m}}{t-\tau_m}\pm\frac{e^{-2\pi a\tau_m}}{t+\tau_m}\right]dt = 0$$

with τ_m defined in (2.26), and further

$$\fint_0^\infty \hat{\varphi}_0(it)\sin(2\pi t)\,e^{-2\pi t/\beta}\left[\frac{e^{2\pi\nu_m/\beta}}{t-\nu_m}\pm\frac{e^{-2\pi\nu_m/\beta}}{t+\nu_m}\right]dt = 0,$$

where $\hat{\varphi}_0(k) = \mathcal{F}[\chi_B\varphi](k) \in PW^{1/\beta}$, and

$$\nu_m := h\tau_m = \pm\frac{\arccos\lambda}{2\pi} + m, \quad m \in \mathbb{Z}. \tag{2.138}$$

Now note that, for $\beta \ll 1$, one of the terms in the square brackets is exponentially smaller than the other[13], and so

$$\fint_0^\infty \hat{\varphi}_0(it)\,e^{-2\pi t/\beta}\sin(2\pi t)\frac{1}{t-\nu_m}dt = \mathcal{O}\left(\exp\left(-\frac{2}{\sqrt{\pi\beta}}\right)\right), \quad \nu_m > 0,$$

and

$$\int_0^\infty \hat{\varphi}_0(it)\,e^{-2\pi t/\beta}\sin(2\pi t)\frac{1}{t+\nu_m}dt = \mathcal{O}\left(\exp\left(-\frac{2}{\sqrt{\pi\beta}}\right)\right), \quad \nu_m < 0,$$

which are approximate vanishing conditions on a set of equally spaced points of Hilbert and Stieltjes half-line transforms of $\hat{\varphi}_0(it)\,e^{-2\pi t/\beta}\sin(2\pi t) \in H(\mathbb{C}) \cap L^2(\mathbb{R})$, respectively.

Asymptotic connection to a hypersingular integral equation

It is worth mentioning the existence of an asymptotic connection of (2.1) with another well-known integral equation. This connection follows from an approximation of the Poisson operator.

Proposition 2.3.4. *Let $f \in W^{3,2}(\mathbb{R})$. Then, for $h \ll 1$,*

$$P_h[f](x) = f(x) - h\mathcal{H}[f'](x) + r(x), \quad x \in \mathbb{R}, \tag{2.139}$$

[13] The given bound is obtained from the estimate $\mathcal{O}\left(\exp\left(-\frac{\arccos\lambda}{\beta}\right)\right) = \mathcal{O}\left(\exp\left(-\frac{1}{\beta}\arccos\left(\frac{2}{\pi}\arctan\frac{1}{\beta}\right)\right)\right)$ using (2.10) and is the worst possible - for $m \neq 0$ it would be $\mathcal{O}\left(\exp\left(-\frac{2\pi m}{\beta}\right)\right)$.

with continuous decaying to zero at infinity function $r(x)$ such that $\|r\|_{L^\infty(\mathbb{R})} = o(h)$, and Hilbert transform operator $\mathcal{H}$ defined as in (2.136).

Proof. Using $\int_{\mathbb{R}} p_h(t)\,dt = 1$, let us write

$$P_h[f](x) = \frac{h}{\pi}\int_{\mathbb{R}} \frac{f(t)\,dt}{(x-t)^2+h^2} = f(x) + \frac{h}{\pi}\int_{\mathbb{R}} \frac{f(t)-f(x)}{(x-t)^2+h^2}dt,$$

and, choosing some small $\delta > 0$, perform a range decomposition followed by the change of variable $\tau = \dfrac{t-x}{\delta}$

$$\begin{aligned}\int_{\mathbb{R}} \frac{f(t)-f(x)}{(x-t)^2+h^2}dt &= \left(\int_{x-\delta}^{x+\delta} + \int_{\mathbb{R}\setminus(x-\delta,x+\delta)}\right)\frac{f(t)-f(x)}{(x-t)^2+h^2}dt \\ &= \delta\int_{-1}^{1}\frac{f(x+\delta\tau)-f(x)}{\delta^2\tau^2+h^2}d\tau + \delta\int_1^\infty \frac{f(x+\delta\tau)+f(x-\delta\tau)-2f(x)}{\delta^2\tau^2+h^2}d\tau \\ &=: I_1(x) + I_2(x).\end{aligned}$$

To estimate $I_1(x)$, we employ Taylor expansion with the residual term written in the integral form

$$f(x+\delta\tau) = f(x) + \delta\tau f'(x) + \frac{\delta^2\tau^2}{2}f''(\theta(x)), \tag{2.140}$$

with some $\theta(x) \in (x, x+\delta\tau)$ in the Lagrange remainder term. Continuity of f'' allowing this application of Taylor theorem is due to Sobolev embedding $W^{1,2}(\mathbb{R}) \subset C(\mathbb{R})$ (Morrey's inequality [12, Sect. 5.6 Thm 4]), and $f'' \in W^{1,2}(\mathbb{R})$ follows from the assumption.

The linear term in the above Taylor expansion integrates to zero by symmetry of the interval, and therefore we have

$$I_1(x) = \frac{\delta}{2}f''(\theta_0(x))\int_{-1}^{1}\frac{\tau^2 d\tau}{\tau^2+h^2/\delta^2} = \delta f''(\theta_0(x))\left(1 - \frac{h}{\delta}\arctan\frac{\delta}{h}\right) \tag{2.141}$$

with some $\theta_0(x) \in [-1, 1]$.

Computations for I_2 are more subtle and will be performed in Fourier domain. Using Fubini theorem after Fourier transform, we obtain

$$\hat{I}_2(k) = \delta\hat{f}(k)\int_1^\infty \left(e^{2\pi ik\delta\tau} + e^{-2\pi ik\delta\tau} - 2\right)\frac{d\tau}{\delta^2\tau^2+h^2}.$$

Note that

$$e^{2\pi ik\delta\tau} + e^{-2\pi ik\delta\tau} - 2 = 2\left(\cos(2\pi k\delta\tau) - 1\right) = -4\sin^2(\pi k\delta\tau),$$

and making use of the expansion

$$\frac{1}{\delta^2\tau^2+h^2} = \frac{1}{\delta^2\tau^2}\frac{1}{1+(h/\delta\tau)^2} = \frac{1}{\delta^2\tau^2}\left(1 - \frac{1}{(1+\theta_1)^2}\frac{h^2}{\delta^2\tau^2}\right)$$

with some $\theta_1 \in \left[0, h^2/\delta^2\right]$, we get

$$\begin{aligned}\hat{I}_2(k) &= -4\delta \hat{f}(k) \int_1^\infty \frac{\sin^2(\pi k \delta \tau)}{\delta^2 \tau^2} \left(1 - \frac{1}{(1+\theta_1)^2} \frac{h^2}{\delta^2 \tau^2}\right) d\tau \\ &= -4\pi |k| \hat{f}(k) \int_{\pi|k|\delta}^\infty \left(\frac{\sin u}{u}\right)^2 \left(1 - \frac{1}{(1+\theta_1)^2} \frac{\pi^2 h^2 k^2}{u^2}\right) du.\end{aligned}$$

Invoking Parseval's theorem and inequality $\dfrac{\sin u}{u} \leq 1$, we compute

$$\int_0^\infty \left(\frac{\sin u}{u}\right)^2 du = \frac{\pi}{4} \int_{\mathbb{R}} \left(\frac{\sin 2\pi u}{\pi u}\right)^2 du = \frac{\pi}{4} \left\|\chi_{(-1,1)}\right\|^2_{L^2(\mathbb{R})} = \frac{\pi}{2}$$

$$\Rightarrow \quad \int_{\pi|k|\delta}^\infty \left(\frac{\sin u}{u}\right)^2 du = \frac{\pi}{2} - \int_0^{\pi|k|\delta} \left(\frac{\sin u}{u}\right)^2 du.$$

Estimating also

$$0 \leq \int_0^{\pi|k|\delta} \left(\frac{\sin u}{u}\right)^2 du \leq \pi |k| \delta, \quad 0 \leq \int_{\pi|k|\delta}^\infty \left(\frac{\sin u}{u}\right)^2 \frac{\pi^2 h^2 k^2}{u^2} du \leq \pi \frac{h^2}{\delta} |k|,$$

we thus conclude

$$\hat{I}_2(k) = -2\pi^2 |k| \hat{f}(k) + \tilde{r}_2(k), \tag{2.142}$$

where

$$\tilde{r}_2(k) = 4\pi |k| \hat{f}(k) \left[\int_0^{\pi|k|\delta} \left(\frac{\sin u}{u}\right)^2 du + \frac{\pi^2 h^2 k^2}{(1+\theta_1)^2} \int_{\pi|k|\delta}^\infty \left(\frac{\sin u}{u}\right)^2 \frac{1}{u^2} du\right] \tag{2.143}$$

$$\Rightarrow \quad |\tilde{r}_2(k)| \leq 4\pi^2 \delta k^2 \left|\hat{f}(k)\right| \left(1 + \frac{1}{(1+\theta_1)^2} \frac{h^2}{\delta^2}\right).$$

Note that since $-2\pi i k \hat{f}(k) = \mathcal{F}[f'](k)$ and $\mathcal{F}[1/x](k) = \pi i \operatorname{sgn} k$, we have

$$\mathcal{F}^{-1}\left[-2\pi^2 |k| \hat{f}(k)\right](x) = \fint_{\mathbb{R}} \frac{f'(t)\, dt}{x - t}.$$

Now it remains to estimate the residue term in (2.139).

Since $|\arctan x| \leq \pi/2$, we deduce that

$$\|r\|_{L^\infty(\mathbb{R})} \leq \frac{h}{\pi} \left[\delta \|f''\|_{L^\infty(\mathbb{R})} \left(1 + \frac{\pi h}{2\delta}\right) + 4\pi^2 \delta \left(1 + \frac{h^2}{\delta^2}\right) \int_{\mathbb{R}} k^2 \left|\hat{f}(k)\right| dk,\right]$$

where the first term in the square brackets comes from (2.141) while the second one is due to a crude estimate of the inverse Fourier transform of $\tilde{r}_2$ in (2.142) done in spirit of Riemann-Lebesgue lemma which also guarantees continuity and decay to zero at infinity.

Finally, choosing $\delta = \mathcal{O}(h^\alpha)$, $\alpha \in (0,1)$ and employing

$$\begin{aligned}\int_{\mathbb{R}} k^2 \left|\hat{f}(k)\right| dk &= \frac{1}{4\pi^2} \left\|\mathcal{F}[f'']\right\|_{L^1(\mathbb{R})} \le \frac{1}{4\pi^2}\left(\int_{\mathbb{R}} \left(1+k^2\right)\left[\mathcal{F}[f''](k)\right]^2\right)^{1/2}\left(\int_{\mathbb{R}} \frac{dk}{1+k^2}\right)^{1/2} \\ &\le C\left\|f''\right\|_{W^{1,2}(\mathbb{R})},\end{aligned}$$

we conclude that $\|r\|_{L^\infty(\mathbb{R})} = \mathcal{O}\left(h^{1+\alpha}\right)$ which finishes the proof. □

Remark 2.3.4. *It is worth noting that the next-order approximation of P_h turns out to be a local operator (second derivative). Indeed, under stronger regularity assumption $f \in W^{5,2}(\mathbb{R})$, the approximation result can be extended to*

$$P_h[f](x) = f(x) - h\mathcal{H}[f'](x) - \frac{1}{2}h^2 f''(x) + r(x), \quad x \in \mathbb{R}, \tag{2.144}$$

with $\|r\|_{L^\infty(\mathbb{R})} = o\left(h^2\right)$, as can be shown following the same line of proof.

The second-order differential operator in (2.144) comes from a higher-order Taylor expansion of f in the I_1 part of the decomposition using the approximation $\arctan\frac{\delta}{h} = \frac{\pi}{2} - \frac{h}{\delta} + \mathcal{O}\left(\frac{h^3}{\delta^3}\right)$. This expansion also produces the term $\frac{h\delta}{\pi} f''(x)$ which will be cancelled by its counterpart in I_2, once we use in there a more subtle estimate

$$\int_0^{\pi|k|\delta}\left(\frac{\sin u}{u}\right)^2 du = \int_0^{\pi|k|\delta}\left[\left(\frac{\sin u}{u}\right)^2 - 1\right] du + \pi|k|\delta$$

combined with the inequality $1 - \left(\frac{\sin u}{u}\right)^2 \le \frac{1}{3}u^2$.

Remark 2.3.5. *The idea of construction of an approximate operator (which, in fact, is an infinitesimal generator of the Poisson semigroup) is not new, and an L^2-version of the Proposition under weaker regularity assumptions on f was stated in [27]. This suggests a refinement which is going to be useful.*

Let $AC(\mathbb{R})$ denote the space of absolutely continuous functions on any interval of $\mathbb{R}$.

Lemma 2.3.2. *The conclusion of Proposition 2.3.4 still holds true under the assumptions $f \in W^{2,2}(\mathbb{R})$, $f' \in AC(\mathbb{R})$.*

Proof. The proof follows essentially the same steps as the one of Proposition 2.3.4, though some estimates have to be sharpened.

First of all, in the Taylor expansion (2.140), using absolute continuity assumption $f' \in AC(\mathbb{R})$, we rewrite the remainder term in the integral form

$$f(x+\delta\tau) = f(x) + \delta\tau f'(x) + \int_x^{x+\delta\tau}(x+\delta\tau - t) f''(t)\,dt.$$

Then,

$$\left|\int_x^{x+\delta\tau}(x+\delta\tau-t) f''(t)\,dt\right| = \left|\int_0^{\delta\tau} t f''(x+\delta\tau-t)\,dt\right| \le \frac{1}{\sqrt{3}}\delta^{3/2}|\tau|^{3/2}\left\|f''\right\|_{L^2(\mathbb{R})},$$

and hence

$$|I_1(x)| \leq \frac{2}{\sqrt{3}} \|f''\|_{L^2(\mathbb{R})} \delta^{1/2} \int_0^1 \frac{\tau^{3/2} d\tau}{\tau^2 + h^2/\delta^2} \leq \frac{4}{\sqrt{3}} \|f''\|_{L^2(\mathbb{R})} \delta^{1/2}. \tag{2.145}$$

More subtle estimate can be obtained for the I_2 part as well.

Denoting the first and the second terms in (2.143) as $\tilde{r}_2^{(1)}(k)$, $\tilde{r}_2^{(2)}(k)$, respectively, we start with the second one and obtain a better bound by making a change of variable $v = \frac{1}{u}$ in the integral

$$\tilde{r}_2^{(2)}(k) := \frac{4\pi^3 h^2}{(1+\theta_1)^2} |k|^3 \hat{f}(k) \int_{\pi|k|\delta}^{\infty} \frac{\sin^2 u}{u^4} du = \frac{4\pi^3 h^2}{(1+\theta_1)^2} |k|^3 \hat{f}(k) \int_0^{\frac{1}{\pi|k|\delta}} v^2 \sin^2(1/v)\, dv$$

$$\Rightarrow \quad \left|\tilde{r}_2^{(2)}(k)\right| \leq \frac{4h^3}{3\delta^3} \left|\hat{f}(k)\right| \quad \Rightarrow \quad \left\|\mathcal{F}^{-1}\left[\tilde{r}_2^{(2)}\right]\right\|_{L^\infty(\mathbb{R})} \leq \frac{4h^3}{3\delta^3} \left\|\hat{f}\right\|_{L^1(\mathbb{R})} \leq \frac{h^3}{\delta^3} C \|f\|_{W^{1,2}(\mathbb{R})}. \tag{2.146}$$

As far as the contribution of the first term in (2.143) is concerned, we do not require now its absolute integrability but instead we estimate its Fourier transform directly using the convolution theorem

$$\begin{aligned}
\mathcal{F}^{-1}\left[\tilde{r}_2^{(1)}\right](x) &= \int_{\mathbb{R}} e^{-2\pi i k x} 4\pi |k| \hat{f}(k) \int_0^{\pi|k|\delta} \left(\frac{\sin u}{u}\right)^2 du dk \\
&= -\int_{\mathbb{R}} f''(x-t)\, \mathcal{F}\left[\frac{1}{\pi|k|} \int_0^{\pi|k|\delta} \left(\frac{\sin u}{u}\right)^2 du\right](t)\, dt
\end{aligned}$$

We employ Parseval's identity and square integrability of the function $\frac{1}{x}\int_0^x \left(\frac{\sin u}{u}\right)^2 du$

$$\begin{aligned}
\int_{\mathbb{R}} \left(\mathcal{F}\left[\frac{1}{\pi|k|} \int_0^{\pi|k|\delta} \left(\frac{\sin u}{u}\right)^2 du\right]\right)^2 (t)\, dt &= \int_{\mathbb{R}} \left(\frac{1}{\pi|k|} \int_0^{\pi|k|\delta} \left(\frac{\sin u}{u}\right)^2 du\right)^2 dk \\
&= \frac{2\delta}{\pi} \int_0^\infty \left(\frac{1}{\kappa} \int_0^\kappa \left(\frac{\sin u}{u}\right)^2 du\right)^2 d\kappa,
\end{aligned}$$

and so, by Cauchy-Schwarz inequality, we arrive at the estimate

$$\left\|\mathcal{F}^{-1}\left[\tilde{r}_2^{(1)}\right]\right\|_{L^\infty(\mathbb{R})} \leq C\delta^{1/2} \|f''\|_{L^2(\mathbb{R})}. \tag{2.147}$$

Finally, taking $\delta = \mathcal{O}(h^\alpha)$, $\alpha \in (0,1)$, and using estimates (2.145), (2.146), (2.147), we conclude that

$$\begin{aligned}
\|r\|_{L^\infty(\mathbb{R})} &\leq \frac{h}{\pi}\left[\|I_1\|_{L^\infty(\mathbb{R})} + \left\|\mathcal{F}^{-1}\left[\tilde{r}_2^{(1)}\right]\right\|_{L^\infty(\mathbb{R})} + \left\|\mathcal{F}^{-1}\left[\tilde{r}_2^{(2)}\right]\right\|_{L^\infty(\mathbb{R})}\right] \\
&= \mathcal{O}\left(h^{1+\alpha/2}\right).
\end{aligned}$$

□

To make use of the obtained approximation result, we first rewrite (2.1) as

$$P_h[\chi_A(f-g)](x) = \lambda(f(x) - g(x)) + \lambda g(x) - P_h[\chi_A g](x), \quad x \in A,$$

where $g \in W^{3,2}(A)$ is any function chosen such that $g^{(n)}(\pm a) = f^{(n)}(\pm a)$, $n = 0, 1$, where the derivatives of f at the end points $x = \pm a$ are known to exist due to Proposition 2.1.1 with yet *a priori* unknown values.

Note that $\tilde{f} := \chi_A (f - g) \in W^{2,2}(\mathbb{R})$ and, since $\tilde{f}''$ exists almost everywhere, it is also true that $\tilde{f}' \in AC(\mathbb{R})$, altogether making Lemma 2.3.2 applicable. This yields an approximate non-homogeneous integral equation

$$\frac{h}{\pi} \fint_A \frac{\tilde{f}'(t)\, dt}{x - t} = (1 - \lambda) \tilde{f}(x) + \tilde{g}(x), \quad x \in A, \tag{2.148}$$

where $\tilde{g}(x) := \lambda g(x) - P_h [\chi_A g](x)$.

We thus obtained what is known as the Prandtl lifting line equation arising in the contexts of hydrodynamics [10] and fracture mechanics [46]. It is sometimes rewritten, integrating by parts, in a hypersingular form in terms of Hadamard finite part integral $\fint_A \dfrac{\tilde{f}(t)\, dt}{(x-t)^2}$.

Equation (2.148) has been studied for decades (see e.g. [11] with a link to Cauchy random processes). The unique solution is known to exist, but, apparently, no closed form of it is known so far. Nevertheless, it can be solved numerically by means of reduction to a homogeneous integral equation with a symmetric regular (so-called Betz) kernel[14] [27]

$$\log \frac{a^2 - xt + \sqrt{(a^2 - x^2)(a^2 - t^2)}}{a^2 - xt - \sqrt{(a^2 - x^2)(a^2 - t^2)}} = 4 \sum_{m=1}^{\infty} \frac{\sin\left(m \arccos \frac{x}{a}\right) \sin\left(m \arccos \frac{t}{a}\right)}{m}$$

followed by efficient expansion of its solution in terms of Chebyshev polynomials [67, 68].

We also note that, although formal asymptotic solution for (2.148) is available for small values of h in [17, 73], the present case is more subtle since the non-integral term in the right-hand side also tends to zero with h according to (2.11).

Once the equation (2.148) is solved, the solution in terms of still unknown constants λ, $f^{(n)}(\pm a)$, $n = 0, 1$ have to be plugged into the original equation (2.1) to yield a homogeneous system of four (which, by parity, reduces to two if the interpolating function g was chosen according to the parity of f) linear algebraic equations

$$-\frac{h}{\pi} \int_{-a}^{a} f(t) \frac{\partial^n}{\partial t^n} \frac{1}{(a \mp t)^2 + h^2} dt = \lambda f^{(n)}(\pm a), \quad n = 0, 1.$$

Equating to zero determinant of this system produces a characteristic equation for λ. Insertion of λ and corresponding set of values $f^{(n)}(\pm a)$, $n = 0, 1$ back in approximate solution results in approximation of a respective eigenfunction.

2.4 Numerical illustrations

In this section we compare the obtained asymptotical results for both cases $\beta \gg 1$ and $\beta \ll 1$ with direct numerical solutions of the integral equation (2.1). It is convenient to obtain the latter, by working with the scaled version of

[14] This kernel arises after inversion of the Cauchy integral and further integration.

the problem with the p_β kernel (recall (2.3)) on the interval $(-1,1)$. Reducing the problem to (2.12), we discretize the integral operator using $N=100$ points Gauss-Legendre quadrature rule

$$\int_{-1}^{1} p_\beta\left(x-t\right)\phi\left(t\right)dt \simeq \sum_{j=1}^{N}\omega_j p_\beta\left(x-t_j\right)\phi_j,$$

where $\phi_j := \phi\left(t_j\right)$, the points $\{t_j\}_{j=1}^N$ are chosen to be the roots of the N-th Legendre polynomial P_N, and the quadrature weights are given by

$$\omega_j := \frac{2\left(1-t_j^2\right)}{N^2 P_{N-1}^2\left(t_j\right)}, \quad j=1,\,\ldots,\,N.$$

Evaluation of the discretized version of the problem

$$\sum_{j=1}^{N}\omega_j p_\beta\left(x-t_j\right)\phi_j = \lambda\phi\left(x\right), \quad x\in\left(-1,1\right), \tag{2.149}$$

at each of the points $\{t_j\}_{j=1}^N$ leads to a set of equations

$$\sum_{j=1}^{N} p_\beta\left(t_i-t_j\right)\omega_j\phi_j = \lambda\phi_i, \quad i=1,\,\ldots,\,N, \tag{2.150}$$

which we solve to find the eigenvalues $\{\lambda_n\}_{n=1}^N$ and the values of the respective eigenfunctions $\phi_j^{[n]}$ at the discretization points $\{t_j\}_{j=1}^N$. Using these values, we reconstruct the eigenfunctions from (2.149) as

$$\phi^{[n]}\left(x\right) = \frac{1}{\lambda_n}\sum_{j=1}^{N}\omega_j\phi_j^{[n]}p_\beta\left(x-t_j\right), \quad n=1,\,\ldots,\,N. \tag{2.151}$$

Eigenfunctions and eigenvalues computed this way with normalization[15] $\left\|\phi^{[n]}\right\|_{L^2(-1,1)}=1$ will be referenced as numerical solution in all further comparisons.

Case $\beta \gg 1$:

On Figures 2.4.1-2.4.6 we compare first few eigenfunctions with prolate spheroidal wave functions $S_{0n}\left(\dfrac{\sqrt{6}}{\beta},x\right)$ which satisfy ODE (2.37) asymptotically equivalent to the integral equation (2.12).

For computation of prolate spheroidal wave functions, we use the Fortran code provided in [77] which has been converted into a MATLAB program with f2matlab[16].

We normalize solutions $S_{0n}\left(\dfrac{\sqrt{6}}{\beta},x\right)$ such that $\|S_{0n}\|_{L^2(-1,1)}=1$.

Once solutions to (2.37) are found, they are plugged back into (2.12) to yield respective eigenvalues: $\lambda_n = \left\langle P_\beta\left[S_{0n}\right], S_{0n}\right\rangle_{L^2(-1,1)}$.

Though better asymptotical approximation to eigenfunctions is furnished by solutions of (2.32) rather than (2.37), we observe excellent agreement with numerical solution for first few eigenfunctions: curves are almost indistinguishable so we also plot their difference.

[15] In case $\beta \ll 1$, eigenfunctions will be normalized after rescaling to the interval $(-1/\beta, 1/\beta)$, see further.

[16] See: https://www.mathworks.com/matlabcentral/fileexchange/6218-computation-of-special-functions/

We would also like to note that calculations of eigenfunctions and eigenvalues of higher index are obstructed by computational difficulties related to smallness of eigenvalues (for large values of β, the operator P_β is very contractive), e.g. for $\beta = 10$, the 6-th eigenvalue is already of order 10^{-12}.

Case $\beta \ll 1$:

In this case, we compare normalized asymptotical solutions of (2.13) with numerical ones obtained by rescaling (2.151) to the interval $(-1/\beta, 1/\beta)$ and normalizing them so that $L^2\left(-\frac{1}{\beta}, \frac{1}{\beta}\right)$ norm is 1.

Contrary to the case $\beta \ll 1$, here our computational strategy requires first to determine eigenvalues, and only after that for each eigenvalue the calculation of corresponding eigenfunction can be done. Recalling that even and odd eigenvalues are the roots of the characteristic equations (2.125) and (2.130), respectively, we plot both left- and right-hand sides of each of these equations in order to find intersection points and thus determine eigenvalues. Figures 2.4.7 and 2.4.8 contain plots representing this in terms of $k_0 = -\dfrac{\log \lambda}{2\pi}$ and spectral parameter λ for even and odd cases, respectively. Plots with respect to k_0, i.e. done in a logarithmic scale, are more illuminating due to the geometric decay of eigenvalues (2.11). When compared with vertical lines corresponding to numerically found eigenvalues, we see that matching worsens for higher index eigenvalues: starting from the 30th eigenfunction, the difference between left/right-hand side crossing points and abscissas of vertical lines becomes clearly visible. It is less obvious (though becomes visible after zooming in) that the matching is slightly worse for the very first eigenvalue λ_1 than for the next ones. This is not entirely surprising since it is in a qualitative accordance with the error term estimates: as we noted after Theorem 2.3.1, the asymptotic approximation deteriorates when λ is close to 1 or 0. To illustrate this peculiarity of the asymptotic approximation, we plot on Figure 2.4.19 the quantity $\left\|P_{1/\beta}\left[\varphi_n\right] - \lambda_n \varphi_n\right\|_{L^2\left(-\frac{1}{\beta}, \frac{1}{\beta}\right)}$ to demonstrate *a posteriori* verification of the solution.

Another tool to estimate quality of the obtained solutions is computation of a set of mutual inner products, namely, Gram matrix. Due to orthogonality of eigenfunctions (recall Proposition 2.1.1) and the chosen normalization, it must coincide with the identity matrix on true solutions. Left plot on Figure 2.4.20 shows gradual deviation from the identity as indices of eigenfunctions increase while numerical results are still rather accurate.

Finally, by plotting even and odd eigenfunctions on Figures 2.4.9-2.4.13 and 2.4.14-2.4.18, respectively, we observe that these solutions are very close to sine and cosine families with a deviation (given by integral terms on the right in (2.122) and (2.126)) being non-negligible in magnitude yet localized near the interval endpoints only for higher index eigenfunctions (see Figures 2.4.13 and 2.4.18).

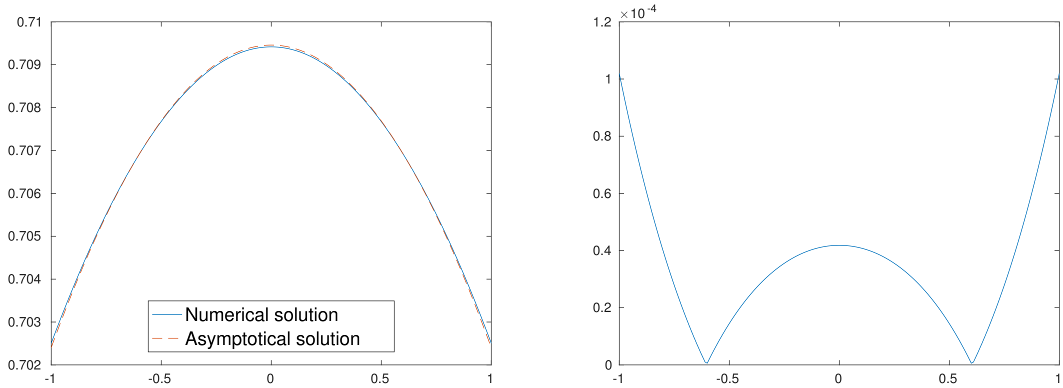

Figure 2.4.1: Comparison of asymptotical and numerical values for $n = 1$ eigenfunction, $\beta = 10$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

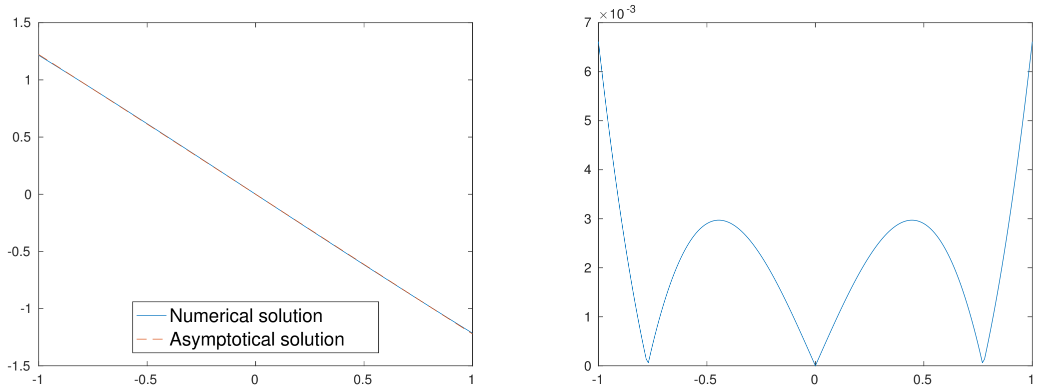

Figure 2.4.2: Comparison of asymptotical and numerical values for $n = 2$ eigenfunction, $\beta = 10$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

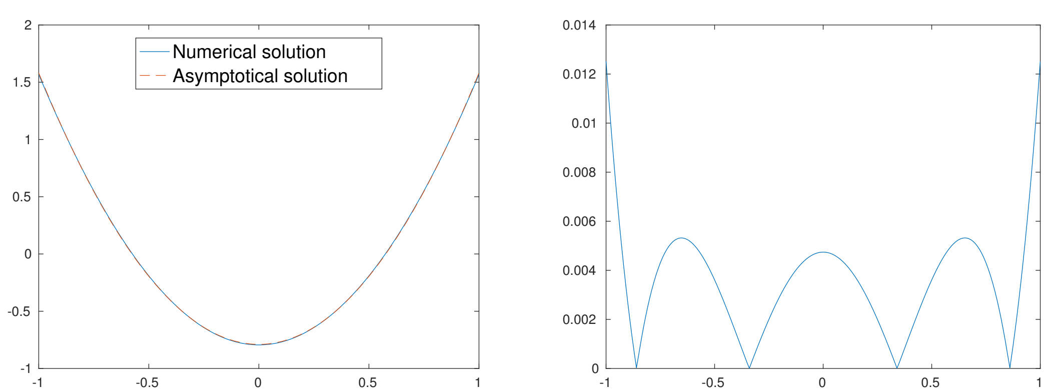

Figure 2.4.3: Comparison of asymptotical and numerical values for $n = 3$ eigenfunction, $\beta = 10$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

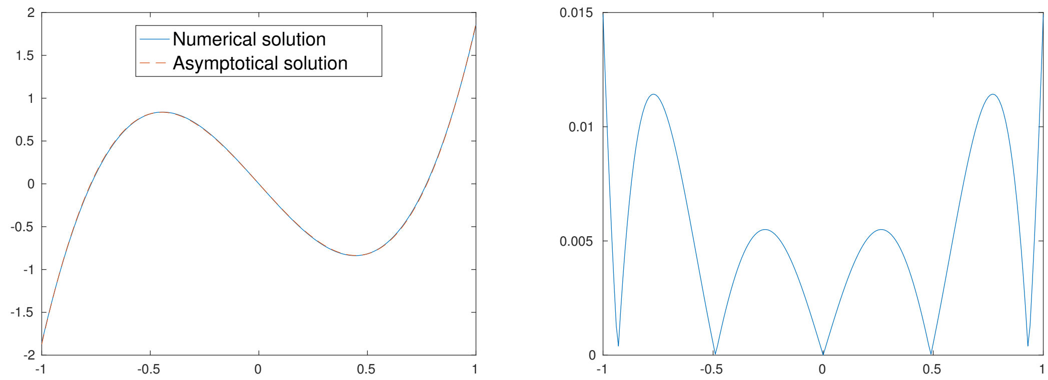

Figure 2.4.4: Comparison of asymptotical and numerical values for $n = 4$ eigenfunction, $\beta = 10$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

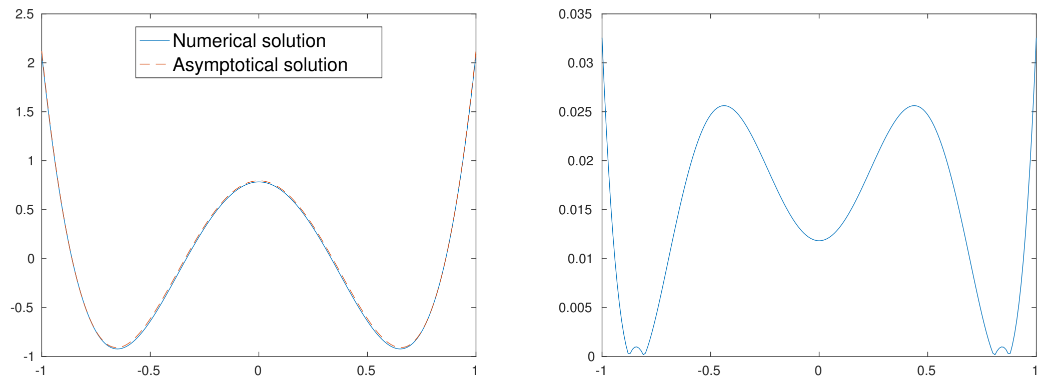

Figure 2.4.5: Comparison of asymptotical and numerical values for $n = 5$ eigenfunction, $\beta = 10$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

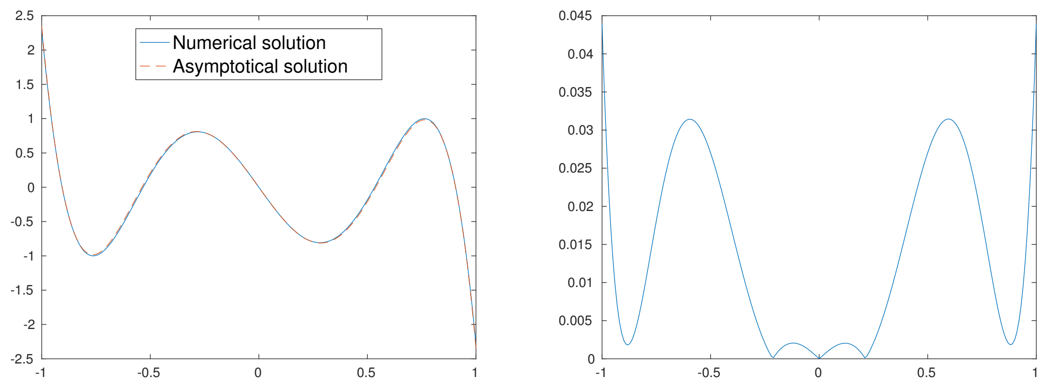

Figure 2.4.6: Comparison of asymptotical and numerical values for $n = 6$ eigenfunction, $\beta = 10$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

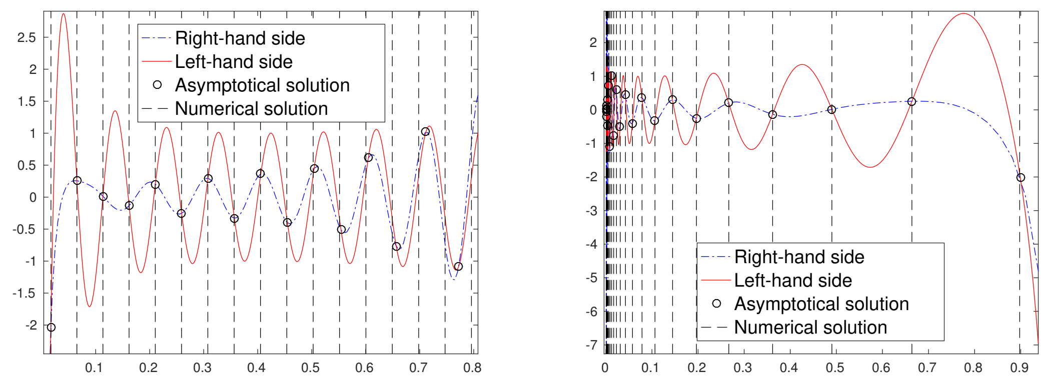

Figure 2.4.7: Computation of eigenvalues corresponding to even eigenfunctions. Left- and right-hand sides of (2.125) and numerical values, $\beta = 0.1$: in terms of k_0 (left) and λ (right)

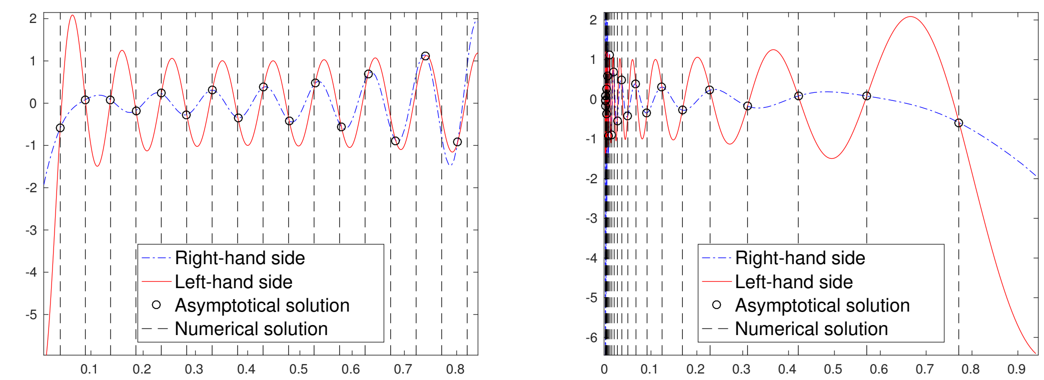

Figure 2.4.8: Computation of eigenvalues corresponding to odd eigenfunctions. Left- and right-hand sides of (2.130) and numerical values, $\beta = 0.1$: in terms of k_0 (left) and λ (right)

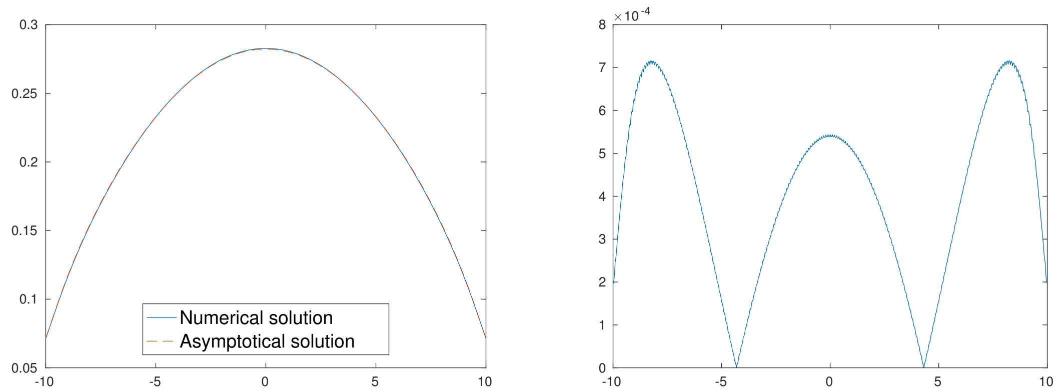

Figure 2.4.9: Comparison of asymptotical and numerical values for $n = 1$ eigenfunction (1st even eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

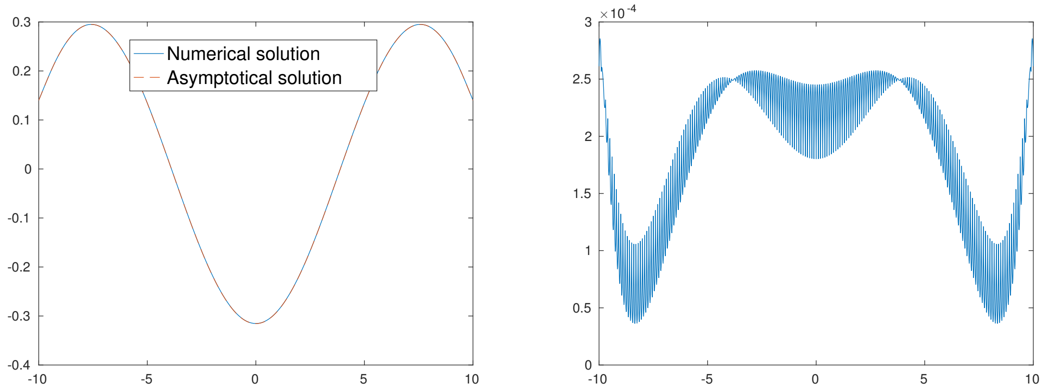

Figure 2.4.10: Comparison of asymptotical and numerical values for $n = 3$ eigenfunction (2nd even eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

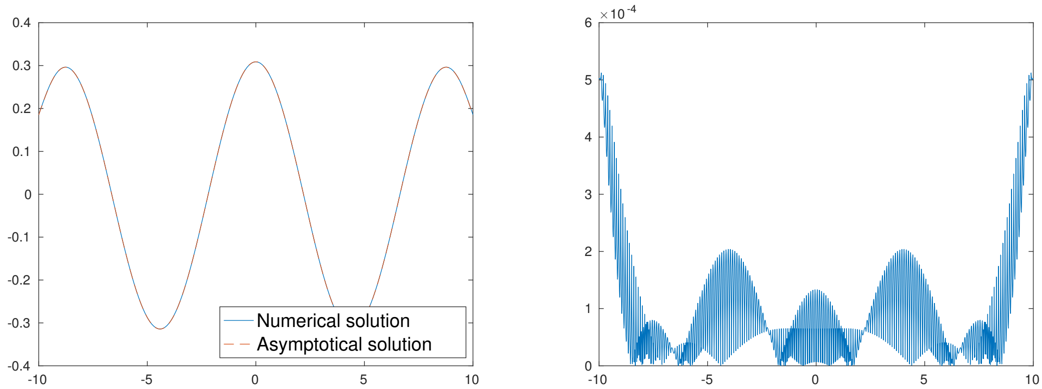

Figure 2.4.11: Comparison of asymptotical and numerical values for $n = 5$ eigenfunction (3rd even eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

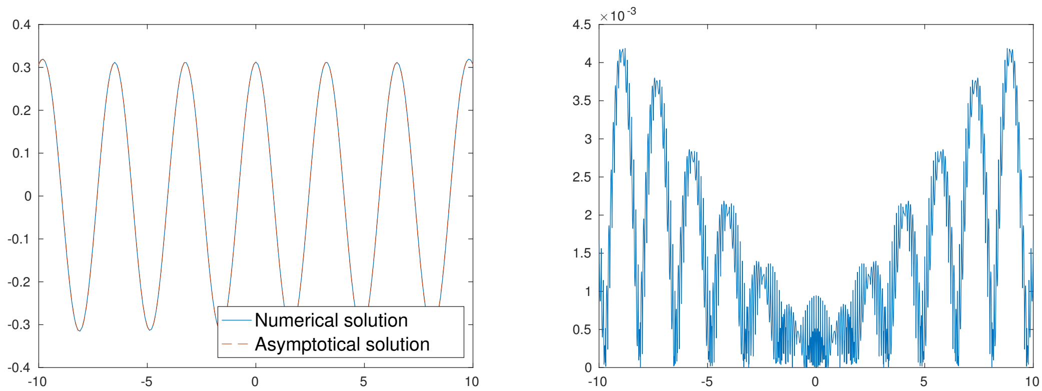

Figure 2.4.12: Comparison of asymptotical and numerical values for $n = 13$ eigenfunction (7th even eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

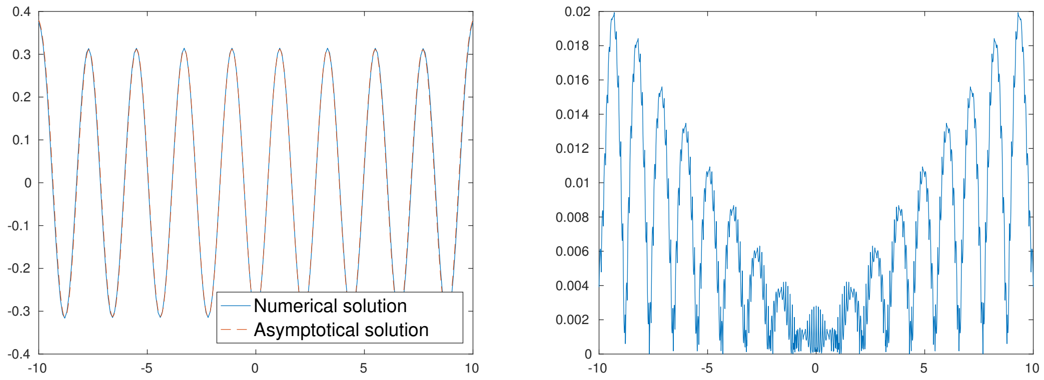

Figure 2.4.13: Comparison of asymptotical and numerical values for $n = 19$ eigenfunction (10th even eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

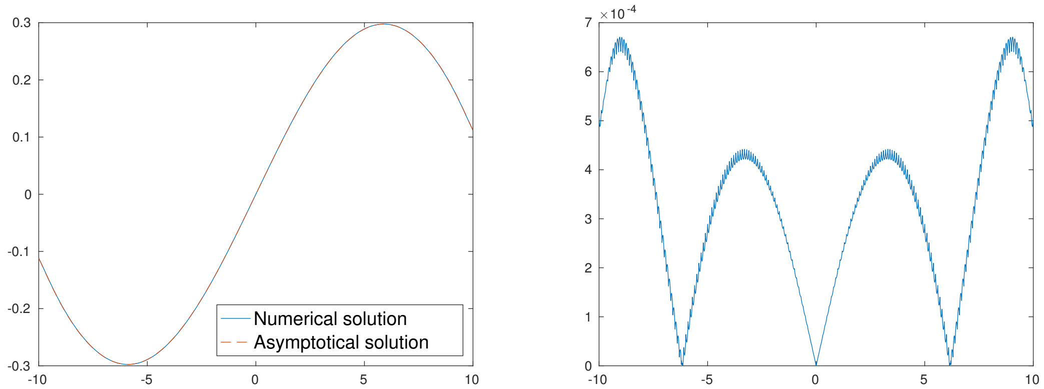

Figure 2.4.14: Comparison of asymptotical and numerical values for $n = 2$ eigenfunction (1st odd eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

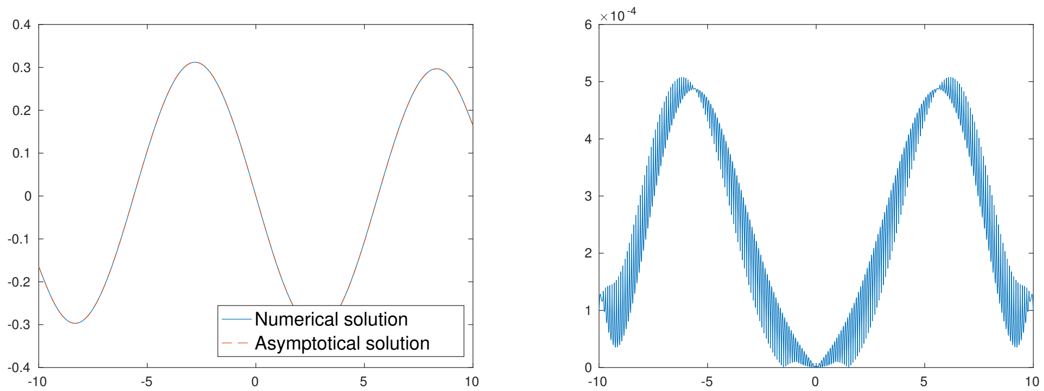

Figure 2.4.15: Comparison of asymptotical and numerical values for $n = 4$ eigenfunction (2nd odd eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

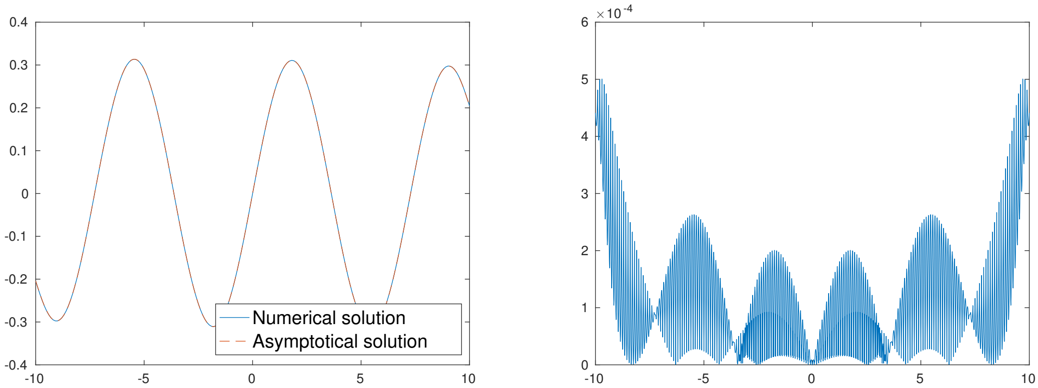

Figure 2.4.16: Comparison of asymptotical and numerical values for $n = 6$ eigenfunction (3rd odd eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

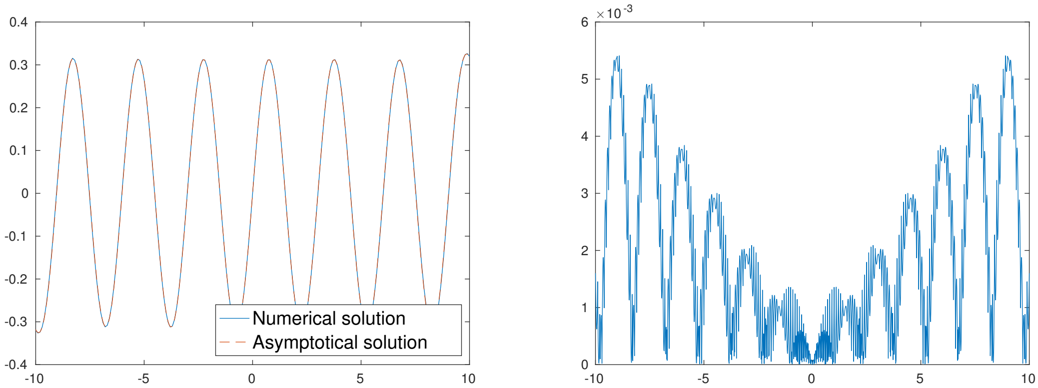

Figure 2.4.17: Comparison of asymptotical and numerical values for $n = 14$ eigenfunction (7th odd eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

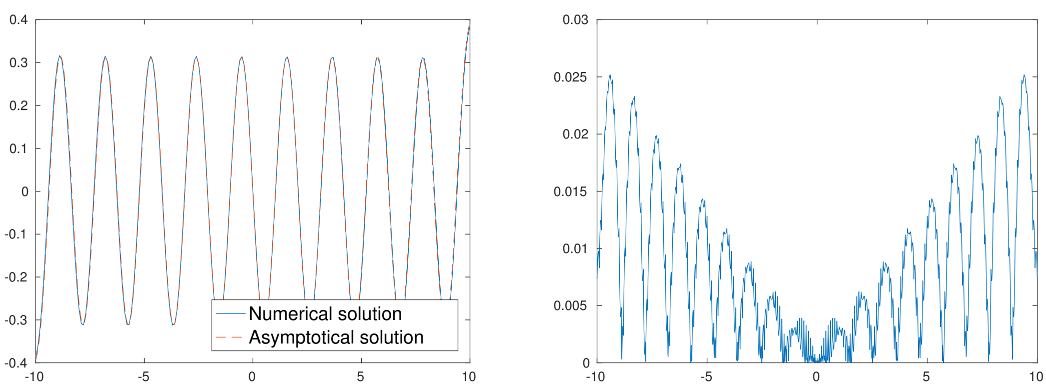

Figure 2.4.18: Comparison of asymptotical and numerical values for $n = 20$ eigenfunction (10th odd eigenfunction), $\beta = 0.1$: eigenfunction values (left) and difference between asymptotical and numerical solutions (right)

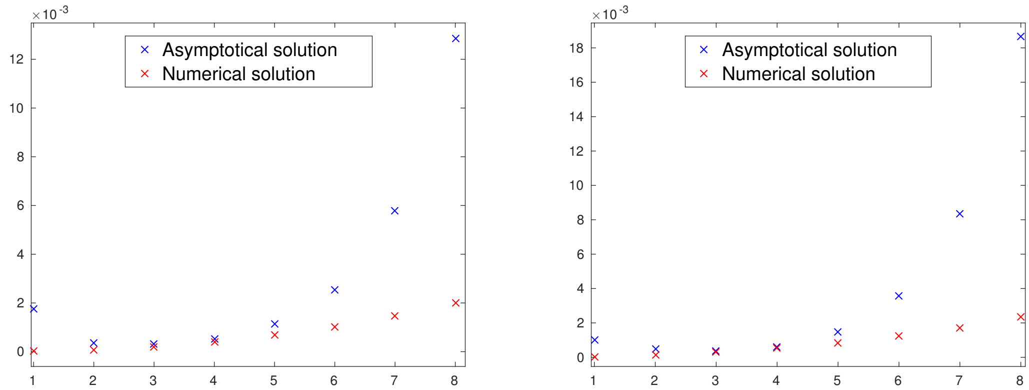

Figure 2.4.19: A posteriori L^2 error estimate for first 16 eigenfunctions: first 8 even (left) and first 8 odd (right)

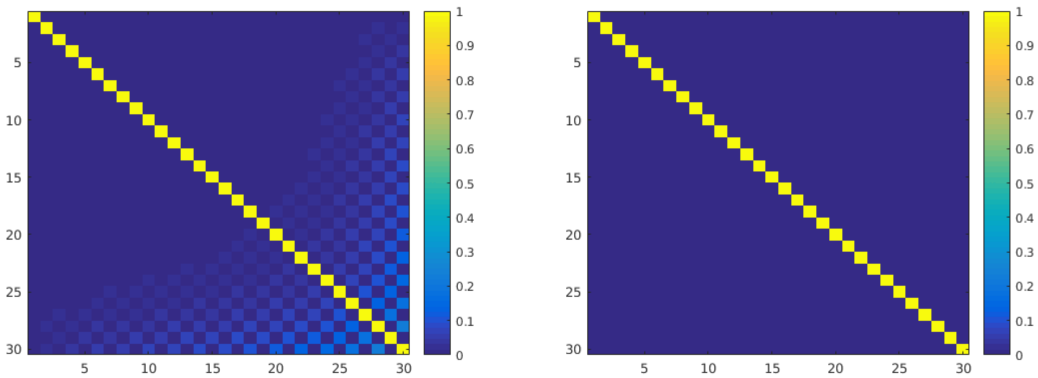

Figure 2.4.20: Shape of the matrix of inner products of first 30 eigenfunctions

APPENDIX

Lemma 2.4.1. *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be an even function such that $t^2 f(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then:*

$$\frac{1}{\pi} \fint_{\mathbb{R}} \frac{f(t)}{t-x} dt = -\frac{2}{\pi x} \int_0^\infty f(t)\, dt + \mathcal{O}\left(\frac{1}{x^3}\right).$$

Proof. Using convolution theorem for Fourier transforms, we can write

$$\frac{i}{\pi} \fint_{\mathbb{R}} \frac{f(t)}{t-x} dt = \mathcal{F}^{-1}\left[\operatorname{sgn} y\, \mathcal{F}[f](y)\right](x) = -4i \int_0^\infty \sin(2\pi xy) \int_0^\infty \cos(2\pi yt)\, f(t)\, dt dy,$$

where we can also employed the even parity of f.

Now, upon double integration by parts (involving differentiation under integral sign which is possible during absolute integrability), it follows that

$$\begin{aligned} \fint_{\mathbb{R}} \frac{f(t)}{t-x} dt &= -4\pi \left[\frac{1}{2\pi x} \int_0^\infty f(t)\, dt - \frac{1}{x} \int_0^\infty \cos(2\pi xy) \int_0^\infty \sin(2\pi yt)\, t f(t)\, dt dy\right] \\ &= -\frac{2}{\pi x} \int_0^\infty f(t)\, dt - \frac{4\pi}{x^2} \int_0^\infty \sin(2\pi xy) \int_0^\infty \cos(2\pi yt)\, t^2 f(t)\, dt dy. \end{aligned}$$

It remains only to notice that since $t^2 f(t) \in L^2(\mathbb{R})$, by isometry of Fourier transform,

$$\int_0^\infty \sin(2\pi xy) \int_0^\infty \cos(2\pi yt)\, t^2 f(t)\, dt dy \in L^2(\mathbb{R}) \quad \Rightarrow \quad \int_0^\infty \sin(2\pi xy) \int_0^\infty \cos(2\pi yt)\, t^2 f(t)\, dt dy = \mathcal{O}\left(\frac{1}{x}\right).$$

$\square$

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PART 3

Recovery of magnetization features by means of Kelvin transformations and Fourier analysis

3.1 Introduction

Some ancient rocks and meteorites possess remanent magnetization and thus might preserve valuable records of the past magnetic field on Earth and other planets, asteroids, and satellites. Thanks to the advances in magnetometry (e.g. SQUID microscopy technique) offering the possibility of measuring magnetic fields of very low intensities with high spatial resolution, extraction of this relict magnetic information has become reality. Deducing magnetization of a geosample hinges on processing the measurements of the weak magnetic field available in its nearest neighborhood. An endeavor to develop a robust and efficient method for processing these data leads to a number of challenging problems such as effective extension of the restricted measurement data and extraction of certain features of the magnetization (typically, its mean value over the whole sample) without solving the entire inverse problem. In particular, we are concerned with study of the following set-up.

Suppose there is a localized sample whose magnetization distribution is described by vector function

$$\vec{\mathcal{M}}\left(\vec{x}\right) \equiv \left(\mathcal{M}_1\left(x_1, x_2, x_3\right), \mathcal{M}_2\left(x_1, x_2, x_3\right), \mathcal{M}_3\left(x_1, x_2, x_3\right)\right)^T$$

with compact support $Q \subset \mathbb{R}^3$.

Choosing the origin in the geometrical center of Q, we define its "diameter" $d_Q := \sup\limits_{\vec{x}, \vec{y} \in Q} |\vec{x} - \vec{y}|$ and height $h_Q := 2 \max\limits_{\vec{x} \in Q} |x_3|$.

The magnetic field $\vec{B}\left(\vec{x}\right)$ produced by the magnetized sample outside its support can be expressed as $\vec{B}\left(\vec{x}\right) =$

$-\nabla\Phi(\vec{x})$, where the scalar potential $\Phi(\vec{x})$ satisfies, in a distributional sense, the Poisson equation [3]

$$\Delta\Phi(\vec{x}) = \nabla\cdot\vec{\mathcal{M}}(\vec{x}), \quad \vec{x}\in\mathbb{R}^3,$$

and hence is given by [4, Sect. 2.4 Thm 1]

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\iiint_Q \frac{1}{\left|\vec{x}-\vec{t}\right|}\nabla\cdot\vec{\mathcal{M}}\left(\vec{t}\right)d^3t.$$

Consequently, performing integration by parts and assuming vanishing of $\vec{\mathcal{M}}(\vec{x})$ on the boundary ∂Q, we obtain, for $\vec{x}\notin Q$,

$$\Phi(\mathbf{x},x_3) = \frac{1}{4\pi}\iiint_Q \frac{\mathcal{M}_1(\mathbf{t},t_3)(x_1-t_1)+\mathcal{M}_2(\mathbf{t},t_3)(x_2-t_2)+\mathcal{M}_3(\mathbf{t},t_3)(x_3-t_3)}{\left(|\mathbf{x}-\mathbf{t}|^2+(x_3-t_3)^2\right)^{3/2}}d^3t, \tag{3.1}$$

where we adopted bold symbols to denote $\mathbb{R}^2$ vectors, e.g. $\mathbf{x}\equiv(x_1,x_2)^T$, a notation that will be convenient throughout the work.

Fixing $x_3 = h > h_Q/2$ defining a horizontal plane that we will refer as the measurement plane, we now arrive at one version of the problem that we are going to study: given $\Phi(\mathbf{x},h)$ for $\mathbf{x}\in T$, where either $T=\mathbb{R}^2$ or $T = D_A := \left\{\vec{x}\in\mathbb{R}^2 : x_1^2+x_2^2 < A^2\right\}$, we want to gain some knowledge on the magnetization of the sample[1], namely, algebraic moments of the magnetization distribution such as

$$\left\langle \mathcal{M}_i x_{k_1}^{j_1}x_{k_2}^{j_2}\right\rangle := \iiint_Q \mathcal{M}_i(\vec{x})\,x_{k_1}^{j_1}x_{k_2}^{j_2}d^3x, \quad i,\,k_1,\,k_2\in\{1,2,3\}, \quad j_1,\,j_2\in\{0,1,2\}. \tag{3.2}$$

The most interesting quantity from a physical point of view, aside from the magnetization itself, is the net moment of the sample, i.e. vector (3.2) with $j_1=j_2=0$, that we also denote as $\vec{m}=(m_1,m_2,m_3)^T$ and refer to m_1, m_2 as tangential and to m_3 as the normal component of the net moment.

In practice, however, instead of the potential data, the measurements are available for the vertical component (normal) of the magnetic field on the horizontal plane $x_3=h$

$$B_3(\mathbf{x},h) = -\partial_{x_3}\Phi(\mathbf{x},h), \tag{3.3}$$

that is, explicitly,

$$B_3(\mathbf{x},h) = \frac{1}{4\pi}\iiint_Q \frac{3(h-t_3)\left[\mathcal{M}_1(\mathbf{t},t_3)(x_1-t_1)+\mathcal{M}_2(\mathbf{t},t_3)(x_2-t_2)\right]+\mathcal{M}_3(\mathbf{t},t_3)\left(2(h-t_3)^2-|\mathbf{x}-\mathbf{t}|^2\right)}{\left(|\mathbf{x}-\mathbf{t}|^2+(h-t_3)^2\right)^{5/2}}d^3t. \tag{3.4}$$

This leads to another version of the problem: computing magnetization moments from knowledge of the left-hand side on T (again, either $T=\mathbb{R}$ or $T=D_A$).

We would like to stress that despite the particular context of paleomagnetism, the formulation given above is

[1] It is clear that reconstructing magnetization distribution $\vec{\mathcal{M}}(\vec{x})$ without additional assumptions on its form is impossible due to the severe ill-posedness of the problem: for example, it is clear that one can add any divergence-free source to $\vec{\mathcal{M}}(\vec{x})$ without changing the potential $\Phi(\mathbf{x},h)$.

a rather general inverse source recovery problem where, based on partial measurements of a harmonic field, some features of the source have to be reconstructed.

It is known [6] that the dipole moment of the source can be obtained from knowledge of potential or field on a sphere surrounding the sources by means of integration of data against first spherical harmonics. As we shall see immediately, this dipole moment is exactly the net moment of the sample.

Indeed, consider, for instance, a hypothetical situation when we have measurements of the potential on a sphere $\mathbb{S}_{R_0}$ encompassing the sample support Q. In spherical coordinates $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$, (3.1) is rewritten as

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi}\iiint_Q \frac{\mathcal{M}_1(\mathbf{t},t_3)(r\sin\theta\cos\phi - t_1) + \mathcal{M}_2(\mathbf{t},t_3)(r\sin\theta\sin\phi - t_2) + \mathcal{M}_3(\mathbf{t},t_3)(r\cos\theta - t_3)}{\left(r^2 - 2r\left[(t_1\cos\phi + t_2\sin\phi)\sin\theta + t_3\cos\theta\right] + t_1^2 + t_2^2 + t_3^2\right)^{3/2}} d^3t. \tag{3.5}$$

Since $\Phi(r,\theta,\phi)$ is harmonic for $r > R_0$, we can expand it over solid harmonics

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\sum_{j=-l}^{l} c_{j,l} S_l^j(\theta,\phi), \quad S_l^j(\theta,\phi) := \begin{cases} P_l^j(\cos\theta)\cos(j\phi), & j \geq 0, \\ P_l^{|j|}(\cos\theta)\sin(|j|\phi), & j < 0, \end{cases} \tag{3.6}$$

where S_l^j are spherical harmonics and P_l^j are associated Legendre polynomials [4].

It is easy to see that the potential (3.5) decays at infinity as $\mathcal{O}(1/r^2)$. This implies that $c_{0,0} = 0$. On the other hand, by orthogonality of spherical harmonics[2], we observe that

$$\lim_{R\to\infty}\left\langle \Phi, \left(S_1^{-1}, S_1^0, S_1^1\right)^T \right\rangle_{L^2(\mathbb{S}_R)} = \left(-\frac{1}{3}m_2, \frac{1}{3}m_3, -\frac{1}{3}m_1\right)^T = \left(\frac{4\pi}{3}c_{-1,1}, \frac{4\pi}{3}c_{0,1}, \frac{4\pi}{3}c_{1,1}\right)^T,$$

which allows us to retrieve the net moment components:

$$m_1 = -3\left\langle \Phi, S_1^1\right\rangle_{L^2(\mathbb{S}_{R_0})}, \quad m_2 = -3\left\langle \Phi, S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})}, \quad m_3 = 3\left\langle \Phi, S_1^0\right\rangle_{L^2(\mathbb{S}_{R_0})}.$$

In this work, we demonstrate to which extent and how one can adapt such spherical geometry methods for the situation when data are available on a plane. To this effect, we introduce specially designed Kelvin transformations to perform data mapping to a sphere of large radius where it is easier to access moment information by means of evaluation of certain asymptotic projections onto first spherical harmonics. This, in particular, produces formulas for the tangential components of net moment in terms of potential or field data available on the whole plane, whereas, interestingly enough, the same strategy is not applicable for the normal component. The latter, however, still can be obtained by different approach based on Poisson representation formula for the ball. The obtained expressions for the tangential and normal components of net moment are of different kind: tangential components are given in terms of integrals of the field over the entire plane while the normal one is a limit of integrals of the field over large circles.

[2]We recall explicit form of the spherical harmonics in question: $S_1^{-1}(\theta,\phi) = \sin\theta\cos\phi$, $S_1^1(\theta,\phi) = \sin\theta\sin\phi$, $S_1^0(\theta,\phi) = \cos\theta$.

We then treat a more practical case when the data are available only on a subset of the plane, namely the disk[3] D_A, and study also what can be achieved with Fourier analysis that takes advantage of convolution structure of the integral operators. In the Fourier domain we devise a systematic way of extraction of moment information by analysing wave vectors in different neighborhoods of the origin.

In passing, we demonstrate an idea that incomplete data can be effectively extended using the fact that asymptotical behavior of the field $B_3(\mathbf{x}, h)$ for large $|\mathbf{x}|$ is proportional to the algebraic moments of the magnetization, exactly the quantities that have to be recovered. In particular, certain linear combinations of integrals result in higher-order formulas for estimating the net moment components. We conclude by showing the effectiveness of these formulas numerically.

Before we embark on the analysis which is going to be tedious, let us make one simplification that will help us reducing the size of intermediate expressions, yet will not affect results for the net moment. Namely, in what follows, we assume the magnetization sample to be planar, i.e. $\vec{\mathcal{M}}(\vec{x}) = \vec{M}(\mathbf{x}) \times \delta(x_3)$, where δ is the one-dimensional Dirac delta function.

Instead of (3.1), (3.4), we therefore consider, respectively,

$$\Phi(\mathbf{x}, h) = \frac{1}{4\pi} \iint_Q \frac{M_1(\mathbf{t})(x_1 - t_1) + M_2(\mathbf{t})(x_2 - t_2) + M_3(\mathbf{t})\, h}{\left(|\mathbf{x} - \mathbf{t}|^2 + h^2\right)^{3/2}} dt_1 dt_2, \tag{3.7}$$

$$B_3(\mathbf{x}, h) = \frac{1}{4\pi} \iint_Q \frac{3h\left[M_1(\mathbf{t})(x_1 - t_1) + M_2(\mathbf{t})(x_2 - t_2)\right] + M_3(\mathbf{t})\left(2h^2 - |\mathbf{x} - \mathbf{t}|^2\right)}{\left(|\mathbf{x} - \mathbf{t}|^2 + h^2\right)^{5/2}} dt_1 dt_2. \tag{3.8}$$

The algebraic moments definition (3.2) is now reduced to the expression

$$\left\langle M_i x_{k_1}^{j_1} x_{k_2}^{j_2} \right\rangle := \iint_Q M_i(\mathbf{x})\, x_{k_1}^{j_1} x_{k_2}^{j_2} dx_1 dx_2, \quad i \in \{1, 2, 3\}, \quad k_1, k_2 \in \{1, 2\}, \quad j_1, j_2 \in \{0, 1, 2\} \tag{3.9}$$

with $\vec{m} = (m_1, m_2, m_3)^T$ still denoting its particular instances for $j_1 = j_2 = 0$.

Obtained results can then be extended back to the three-dimensional case simply by replacing all occurrences of h by $h - t_3$ and integrating further in t_3 variable. This produces more terms in the estimates but neither changes in the analysis nor alternations in the final net moment formulas (3.79)-(3.80), (3.78), (3.90)-(3.91) constituting the main results of the work from the practical point of view. These results are summarized in

Theorem 3.1.1. *Suppose $\vec{\mathcal{M}}$ is a distribution of compact support producing magnetic field whose vertical component is given by (3.4). Then, for $A \gg d_Q$, we have*

$$m_1 = 2\iint_{D_A} x_1 B_3(\mathbf{x}, h)\, dx_1 dx_2 + \mathcal{O}\left(\frac{1}{A}\right) = 2\iint_{D_A} \left(1 + \frac{4x_1^2}{3A^2}\right) x_1 B_3(\mathbf{x}, h)\, dx_1 dx_2 + \mathcal{O}\left(\frac{1}{A^2}\right),$$

$$m_2 = 2\iint_{D_A} x_2 B_3(\mathbf{x}, h)\, dx_1 dx_2 + \mathcal{O}\left(\frac{1}{A}\right) = 2\iint_{D_A} \left(1 + \frac{4x_2^2}{3A^2}\right) x_2 B_3(\mathbf{x}, h)\, dx_1 dx_2 + \mathcal{O}\left(\frac{1}{A^2}\right),$$

[3] This assumption has only computational advantage in estimating the integrals and, in general, can be dropped.

$$m_3 = 2A \iint_{D_A} B_3(\mathbf{x}, h)\, dx_1 dx_2 + \mathcal{O}\left(\frac{1}{A^2}\right).$$

The part is organized as follows. In Section 3.2, we introduce a suitable for our purposes version of Kelvin transformation and give its basic properties that will further be needed. Section 3.3 consists in application of this transform followed by asymptotical analysis yielding formulas for tangential net moment components in terms of potential and field data under assumption of availability of complete measurements. Then, in Section 3.4, by analysis of a different kind, we obtain an asymptotic formula for the normal net moment component in terms of magnetic field. We further extend results to the situation when only partial data are available from measurements, this constitutes the content of Section 3.5. We then show, in Section 3.6, that the same asymptotical formulas can be obtained by different method based on analysis in Fourier domain, moreover, we also obtain improved versions of the formulas estimating tangential net moments components up to higher asymptotical order. Finally, in Section 3.7, we illustrate the obtained results numerically.

3.2 Kelvin transformation

Recall that in the complex plane $\mathbb{C}$, the Moebius transform $\dfrac{z-i}{z+i}$ sends the upper half-plane Im $z > 0$ onto the unit disk $|z| < 1$ preserving harmonicity. Kelvin transformation is a generalization of this concept to higher dimensions.

As discussed in [2], transforms preserving harmonicity are those obtained by translations and reflections with respect to auxiliary spheres or planes. In particular, we consider a transformation that is based on reflection with respect to the auxiliary sphere of radius $e_0 := \sqrt{2R_0(R_0+h)}$ centered at $(0,0,-R_0)$, $R_0 > 0$:

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T \mapsto \mathcal{R}\boldsymbol{\xi} \in \mathbb{R}^3,$$

$$\mathcal{R}\boldsymbol{\xi} := \left(\frac{e_0^2 \xi_1}{\xi_1^2 + \xi_2^2 + (\xi_3 + R_0)^2}, \frac{e_0^2 \xi_2}{\xi_1^2 + \xi_2^2 + (\xi_3 + R_0)^2}, -R_0 + \frac{e_0^2(\xi_3 + R_0)}{\xi_1^2 + \xi_2^2 + (\xi_3 + R_0)^2}\right)^T,$$

which maps the sphere $\mathbb{S}_{R_0}$ of radius R_0 onto the horizontal plane $x_3 = h$ while its south pole $\boldsymbol{s} := (0,0,-R_0)^T$ is mapped to the infinitely far away point. Then, if $f(\mathbf{x}, x_3)$ is a function harmonic in the half-space $x_3 > h$, its Kelvin transform

$$\mathcal{K}[f](\boldsymbol{\xi}) := \frac{1}{|\boldsymbol{\xi} - \boldsymbol{s}|} f(\mathcal{R}\boldsymbol{\xi}) \tag{3.10}$$

defines a harmonic function inside the open ball $\mathbb{B}_{R_0}$ such that $f(\mathbf{x}, h) \mapsto \mathcal{K}[f](\boldsymbol{\xi})|_{\xi \in \mathbb{S}_{R_0}}$ is an isometry:

$$L^2_w(\mathbb{R}^2) := L^2\left(\mathbb{R}^2; \frac{dx_1 dx_2}{x_1^2 + x_2^2 + (R_0+h)^2}\right) \to L^2\left(\mathbb{S}_{R_0}; R_0^2 \sin\theta d\theta d\phi\right) =: L^2(\mathbb{S}_{R_0}),$$

$$\langle f, g \rangle_{L^2_w(\mathbb{R}^2)} = \langle \mathcal{K}[f], \mathcal{K}[g] \rangle_{L^2(\mathbb{S}_{R_0})}, \quad f, g \in L^2_w(\mathbb{R}^2),$$

which is, moreover, an involution, up to an absent factor e_0 in the definition of the transform, i.e. $\mathcal{K}\mathcal{K}[f] = \dfrac{1}{e_0^2} f$.

Straightforward algebraic computations based on the chain rule allow us to establish the following identities valid for $\boldsymbol{\xi} \in \mathbb{S}_{R_0}$:

$$\mathcal{K}\left[\partial_{x_3} f\right](\boldsymbol{\xi}) = -\frac{1}{e_0^2}\left(R_0+\xi_3\right)\left(\mathcal{K}[f](\boldsymbol{\xi}) + 2R_0\partial_r\mathcal{K}[f](\boldsymbol{\xi})\right), \tag{3.11}$$

$$\mathcal{K}\left[x_2\partial_{x_1} f - x_1\partial_{x_2} f\right](\boldsymbol{\xi}) = \xi_2\partial_{\xi_1}\mathcal{K}[f](\boldsymbol{\xi}) - \xi_1\partial_{\xi_2}\mathcal{K}[f](\boldsymbol{\xi}). \tag{3.12}$$

Equation (3.11) is analogous to the one formulated for another Kelvin transform sending interior of the sphere to its exterior and which can be found in [1, 2]. We note the contrast to the two-dimensional situation when normal derivatives are essentially mapped to normal derivatives of transform with no involvement of function itself. The second equation (3.12), in fact, shows commutation of Kelvin transform with the azimuthal angle derivative and consequently suggests the validity of the useful relation

$$\mathcal{K}\left[\int_0^{2\pi} f(\rho\cos\varphi, \rho\sin\varphi, h)\, d\varphi\right](\xi_3) = \left(\int_0^{2\pi}\mathcal{K}[f]\, d\phi\right)(\xi_3), \quad \xi_3 \in (-R_0, R_0), \tag{3.13}$$

where $\rho = \sqrt{x_1^2+x_2^2}$, $\varphi = \arctan\dfrac{x_2}{x_1}$, $\phi = \arctan\dfrac{\xi_2}{\xi_1}$, which indeed holds true, as can be easily checked.

We may also deduce that the transformation $\mathcal{K}$ has its counterpart build upon reflection with respect to the sphere of radius $\tilde{e}_0 := \sqrt{2R_0(R_0-h)}$ centered at $(0,0,R_0)$, $R_0 > h$. Its definition coincides with (3.10) after formal inversion of sign in front of all instances of R_0:

$$\tilde{\mathcal{K}}[f](\boldsymbol{\xi}) := \frac{1}{|\boldsymbol{\xi}+\boldsymbol{s}|} f\left(\tilde{\mathcal{R}}\boldsymbol{\xi}\right), \tag{3.14}$$

$$\tilde{\mathcal{R}}\boldsymbol{\xi} := \left(\frac{\tilde{e}_0^2\xi_1}{\xi_1^2+\xi_2^2+(\xi_3-R_0)^2}, \frac{\tilde{e}_0^2\xi_2}{\xi_1^2+\xi_2^2+(\xi_3-R_0)^2}, R_0 + \frac{\tilde{e}_0^2(\xi_3-R_0)}{\xi_1^2+\xi_2^2+(\xi_3-R_0)^2}\right)^T.$$

Now $\tilde{\mathcal{K}}[f](\boldsymbol{\xi})$ is a function harmonic outside of the ball $\mathbb{B}_{R_0}$ for $f(\mathbf{x}, x_3)$ harmonic in the half-space $x_3 > h$, and the map $f(\mathbf{x}, h) \mapsto \tilde{\mathcal{K}}[f](\boldsymbol{\xi})\Big|_{\xi\in\mathbb{S}_{R_0}}$ is an isometry $L^2_{\tilde{w}}(\mathbb{R}^2) := L^2\left(\mathbb{R}^2; \dfrac{dx_1dx_2}{x_1^2+x_2^2+(R_0-h)^2}\right) \to L^2(\mathbb{S}_{R_0})$.

The mentioned properties of $\mathcal{K}$ remain true also for $\tilde{\mathcal{K}}$ with the exception that (3.11) should be replaced with

$$\tilde{\mathcal{K}}\left[\partial_{x_3} f\right](\boldsymbol{\xi}) = \frac{1}{\tilde{e}_0^2}(R_0-\xi_3)\left(\tilde{\mathcal{K}}[f](\boldsymbol{\xi}) + 2R_0\partial_r\tilde{\mathcal{K}}[f](\boldsymbol{\xi})\right), \quad \boldsymbol{\xi}\in\mathbb{S}_{R_0}. \tag{3.15}$$

However, it is remarkable that $\tilde{\mathcal{K}}$ and the composition transform $\mathcal{K}_0\mathcal{K}$ with $\mathcal{K}_0[f](\boldsymbol{\xi}) := \dfrac{1}{|\boldsymbol{\xi}|} f\left(R_0^2\boldsymbol{\xi}/|\boldsymbol{\xi}|^2\right)$ being reflection with respect to the sphere $\mathbb{S}_{R_0}$ define functions harmonic outside $\mathbb{B}_{R_0}$ which are different. This observation will be constructively used later on.

For $\boldsymbol{\xi}\in\mathbb{S}_{R_0}$, definitions (3.10), (3.14) can be written more explicitly

$$\begin{aligned}
\mathcal{K}[f](\theta,\phi) &= \frac{1}{R_0\sqrt{2(1+\cos\theta)}} f\left(\frac{(R_0+h)\sin\theta\cos\phi}{1+\cos\theta}, \frac{(R_0+h)\sin\theta\sin\phi}{1+\cos\theta}, h\right), && (3.16)\\
\tilde{\mathcal{K}}[f](\theta,\phi) &= \frac{1}{R_0\sqrt{2(1-\cos\theta)}} f\left(\frac{(R_0-h)\sin\theta\cos\phi}{1-\cos\theta}, \frac{(R_0-h)\sin\theta\sin\phi}{1-\cos\theta}, h\right), && (3.17)
\end{aligned}$$

while slightly abusing the notation by writing $\mathcal{K}[f](\theta,\phi)$ in place of $\mathcal{K}[f](R_0\sin\theta\cos\phi, R_0\sin\theta\sin\phi, R_0\cos\theta)$, and similarly for $\tilde{\mathcal{K}}$.

We will also use isometry combined with involution property of both transforms $e_0\mathcal{K}$ and $\tilde{e}_0\tilde{\mathcal{K}}$ in the form of the following identities, for $f_1 \in L^2_w\left(\mathbb{R}^2\right)$, $f_2 \in L^2_{\tilde{w}}\left(\mathbb{R}^2\right)$, $g \in L^2\left(\mathbb{S}_{R_0}\right)$,

$$\left\langle \mathcal{K}\left[f_1\right], g\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = e_0^2 \left\langle f_1, \mathcal{K}\left[g\right]\right\rangle_{L^2_w\left(\mathbb{R}^2\right)}, \quad \left\langle \tilde{\mathcal{K}}\left[f_2\right], g\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = \tilde{e}_0^2 \left\langle f_2, \tilde{\mathcal{K}}\left[g\right]\right\rangle_{L^2_{\tilde{w}}\left(\mathbb{R}^2\right)}. \tag{3.18}$$

3.3 Application for the complete potential or field data

3.3.1 Recovery of tangential components of the net moment

Now let us evaluate (3.16) applied to the potential (3.7)

$$\begin{aligned}\mathcal{K}\left[\Phi\right](\theta,\phi) = & \frac{1}{4\pi R_0\sqrt{2\left(1+\cos\theta\right)}} \iint_Q \left[M_1\left(\mathbf{t}\right)\left(\frac{\left(R_0+h\right)\sin\theta\cos\phi}{1+\cos\theta} - t_1\right)\right. \\ & \left. + M_2\left(\mathbf{t}\right)\left(\frac{\left(R_0+h\right)\sin\theta\sin\phi}{1+\cos\theta} - t_2\right) + M_3\left(\mathbf{t}\right)h\right] \\ & \times \frac{dt_1 dt_2}{\left[\left(\frac{\left(R_0+h\right)\sin\theta\cos\phi}{1+\cos\theta} - t_1\right)^2 + \left(\frac{\left(R_0+h\right)\sin\theta\sin\phi}{1+\cos\theta} - t_2\right)^2 + h^2\right]^{3/2}}.\end{aligned}$$

We are going to integrate this expression against the following spherical harmonics

$$S_1^{-1}(\theta,\phi) = \sin\theta\cos\phi, \quad S_1^1(\theta,\phi) = \sin\theta\sin\phi, \quad S_1^0(\theta,\phi) = \cos\theta,$$

$$S_2^{-1}(\theta,\phi) = \sin\theta\cos\theta\cos\phi, \quad S_2^1(\theta,\phi) = \sin\theta\cos\theta\sin\phi, \quad S_2^0(\theta,\phi) = \cos^2\theta.$$

Denote $\rho_0 := R_0 + h$ and let us start with

$$\begin{aligned}\left\langle \mathcal{K}\left[\Phi\right], S_1^{-1}\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = & \frac{1-h/\rho_0}{4\pi\rho_0} \int_0^\pi \int_0^{2\pi} \frac{\sin^2\theta\cos\phi}{\sqrt{2\left(1+\cos\theta\right)}} \iint_Q \left[M_1\left(\mathbf{t}\right)\left(\frac{\sin\theta\cos\phi}{1+\cos\theta} - \frac{t_1}{\rho_0}\right)\right. \\ & \left. + M_2\left(\mathbf{t}\right)\left(\frac{\sin\theta\sin\phi}{1+\cos\theta} - \frac{t_2}{\rho_0}\right) + M_3\left(\mathbf{t}\right)\frac{h}{\rho_0}\right] \\ & \times \frac{dt_1 dt_2 d\phi d\theta}{\left[\left(\frac{\sin\theta\cos\phi}{1+\cos\theta} - \frac{t_1}{\rho_0}\right)^2 + \left(\frac{\sin\theta\sin\phi}{1+\cos\theta} - \frac{t_2}{\rho_0}\right)^2 + \frac{h^2}{\rho_0^2}\right]^{3/2}}.\end{aligned} \tag{3.19}$$

The key observation is as follows. In the construction of Kelvin transform (3.10) there was a free parameter $R_0 > 0$, the radius of the sphere, where the data were mapped on. In particular, R_0, and hence ρ_0, can be taken as a large number. Utility of this lies in noticing the fact that in the limit $\rho_0 \to \infty$ the denominator simplifies (note that t_1, t_2 range over a bounded set) and the overall expression reduces to the integration of magnetization over Q while integration over ϕ eliminates contribution from M_2 and M_3 terms resulting in an expression proportional only to the net moment component m_1. Nevertheless, such speculation is a little bit naive since the discussed approximation is not uniform in θ due to vanishing of leading terms near $\theta = 0$. The neighborhood of $\theta = \pi$

does not break uniformity since vanishing of $\sin\theta$ is compensated by $(1+\cos\theta)$ in the denominator. Therefore, more careful analysis is required only near $\theta = 0$. We can immediately estimate that the critical neighborhood $\theta = \mathcal{O}(1/\rho_0)$ generally contributes $\mathcal{O}(1/\rho_0)$ to the value of integral which is the same as global contribution from integration over its complementary range in the interval $[0,\pi]$. However, due to orthogonality of trigonometric expressions in ϕ (the dominating term with M_3 factor vanishes), the local contribution to the integral becomes of order $\mathcal{O}\left(1/\rho_0^2\right)$. We are going to see this in detail as we go on to construct asymptotic expansion beyond the leading order aiming to explore possibilities of recovering other algebraic moments of magnetization. We start by splitting the integration range in θ to separate the neighborhood that covers the critical boundary layer $\theta = \mathcal{O}(1/\rho_0)$:

$$\left\langle \mathcal{K}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = \int_0^{\frac{\gamma}{\sqrt{\rho_0}}} (\dots)\, d\theta + \int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi} (\dots)\, d\theta =: I_1 + J_1.$$

We chose the neighborhood size $\mathcal{O}\left(1/\rho_0^{1/2}\right)$ and, solely for book-keeping purposes, we introduced an additional arbitrary constant $0 < \gamma = \mathcal{O}(1)$. It is clear that the final result of matched asymptotic expansion should be independent of these parameters.

We start with I_1 term and rescale by introducing $\theta := \dfrac{\omega}{\rho_0}$ which is a small variable in the range $\left[0, \frac{\gamma}{\sqrt{\rho_0}}\right]$. We employ Taylor expansions

$$\frac{\sin^2\theta}{\sqrt{2(1+\cos\theta)}} \simeq \frac{\theta^2}{2}\left(1 - \frac{5}{24}\theta^2\right), \quad \frac{\sin\theta}{1+\cos\theta} \simeq \frac{\theta}{2}\left(1 + \frac{1}{12}\theta^2\right),$$

as well as expansion for the denominator of (3.19) due to smallness of ω^3/ρ_0^2

$$\frac{1}{\left[\omega^2 - 4\omega(t_1\cos\phi + t_2\sin\phi) + 4\left(t_1^2+t_2^2+h^2\right) + \frac{\omega^3}{6\rho_0^2}\left(\omega - 2(t_1\cos\phi + t_2\sin\phi)\right)\right]^{3/2}}$$

$$\simeq \frac{1}{\left[\omega^2 - 4\omega(t_1\cos\phi + t_2\sin\phi) + 4\left(t_1^2+t_2^2+h^2\right)\right]^{3/2}} - \frac{\omega^3\left(\omega - 2(t_1\cos\phi + t_2\sin\phi)\right)}{4\rho_0^2\left[\omega^2 - 4\omega(t_1\cos\phi + t_2\sin\phi) + 4\left(t_1^2+t_2^2+h^2\right)\right]^{5/2}}$$

to obtain

$$\begin{aligned} I_1 \;\simeq\; & \frac{1}{2\pi\rho_0^4}\left(1 - \frac{h}{\rho_0}\right)\iint_Q \int_0^{2\pi}\left[\rho_0^2\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,\omega^2}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega - \frac{5}{24}\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,\omega^4}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \right. \\ & \left. - \frac{1}{4}\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)(\omega-\alpha)\,\omega^5}{\left[(\omega-\alpha)^2 + c_0\right]^{5/2}} d\omega + \frac{1}{12}\int_0^{\gamma\sqrt{\rho_0}} \frac{c_1\omega^5}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega\right]\cos\phi\, d\phi\, dt_1 dt_2, \end{aligned} \tag{3.20}$$

where

$$\alpha := 2(t_1\cos\phi + t_2\sin\phi), \quad c_0 := 4\left((t_1\sin\phi - t_2\cos\phi)^2 + h^2\right), \tag{3.21}$$

$$c_1 := M_1(\mathbf{t})\cos\phi + M_2(\mathbf{t})\sin\phi, \quad c_2 := -2\left(t_1 M_1(\mathbf{t}) + t_2 M_2(\mathbf{t}) - h M_3(\mathbf{t})\right). \tag{3.22}$$

We deal with the first term in the square brackets in (3.19) by explicit computation of the integral (see

Appendix), whereas for other terms, due to the absence of large factor ρ_0^2, less precise two-term asymptotic expansion is sufficient. This results in:

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,\omega^2}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \simeq c_1\gamma\sqrt{\rho_0} + (3\alpha c_1 + c_2)\log\gamma\sqrt{\rho_0} + c_3 + \frac{3}{\gamma\sqrt{\rho_0}}\left[\frac{1}{2}c_0c_1 - (2\alpha c_1 + c_2)\,\alpha\right],$$

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,\omega^4}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \simeq \frac{1}{3}c_1\gamma^3\rho_0^{3/2} + \frac{1}{2}(3\alpha c_1 + c_2)\,\gamma^2\rho_0,$$

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,(\omega-\alpha)\,\omega^5}{\left[(\omega-\alpha)^2 + c_0\right]^{5/2}} d\omega \simeq \frac{1}{3}c_1\gamma^3\rho_0^{3/2} + \frac{1}{2}(4\alpha c_1 + c_2)\,\gamma^2\rho_0,$$

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{c_1\omega^5}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \simeq \frac{1}{3}c_1\gamma^3\rho_0^{3/2} + \frac{3}{2}\alpha c_1\gamma^2\rho_0,$$

where

$$\begin{aligned} c_3 \quad := \quad & (3\alpha c_1 + c_2)\left(\log\frac{2}{\sqrt{c_0}} + \operatorname{arcsinh}\frac{\alpha}{\sqrt{c_0}}\right) - \frac{1}{c_0}(2\alpha c_1 + c_2)\left[c_0 + \alpha^2 + \alpha\left(c_0 + \alpha^2\right)^{1/2}\right] \\ & -c_1\left[\alpha + \left(c_0 + \alpha^2\right)^{1/2}\right] + \frac{3\alpha^2 c_1 - c_0c_1 + 2\alpha c_2}{c_0\left(c_0 + \alpha^2\right)^{1/2}}\left[c_0 + \alpha^2 + \alpha\left(c_0 + \alpha^2\right)^{1/2}\right], \end{aligned}$$

and we have used the expansion

$$\operatorname{arcsinh} x \simeq \log 2x + \frac{1}{4x^2}, \quad x \gg 1.$$

From definitions (3.21)-(3.22) and orthogonality of trigonometric polynomials, we observe that

$$\int_0^{2\pi} \alpha c_1 \cos\phi d\phi = \int_0^{2\pi} c_2\cos\phi d\phi = 0, \quad \iint_Q \int_0^{2\pi} c_1\cos\phi d\phi dt_1 dt_2 = \pi m_1, \tag{3.23}$$

and hence

$$\int_0^{2\pi}\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,\omega^4}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega\cos\phi d\phi \simeq \int_0^{2\pi}\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,(\omega-\alpha)\,\omega^5}{\left[(\omega-\alpha)^2 + c_0\right]^{5/2}} d\omega\cos\phi d\phi$$

$$\simeq \int_0^{2\pi}\int_0^{\gamma\sqrt{\rho_0}} \frac{c_1\omega^5}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega\cos\phi d\phi \simeq \frac{\pi}{3}M_1(\mathbf{t})\,\gamma^3\rho_0^{3/2}.$$

Therefore,

$$I_1 \quad \simeq \quad \frac{1}{2\pi\rho_0^2}\left[\pi m_1\gamma\sqrt{\rho_0}\left(1 - \frac{\gamma^2 + 8h}{8\rho_0}\right) + \frac{3}{2\gamma\sqrt{\rho_0}}\iint_Q\int_0^{2\pi}\left(c_0c_1 - 4\alpha^2 c_1 - 2\alpha c_2\right)\cos\phi d\phi dt_1 dt_2 + \left(1 - \frac{h}{\rho_0}\right)g_1\right],$$

where

$$g_1 := \frac{1}{\pi}\iint_Q\int_0^{2\pi} c_3\cos\phi d\phi dt_1 dt_2. \tag{3.24}$$

Computation of the quantity

$$
\begin{aligned}
q_1 &:= -\frac{1}{\pi}\iint_Q\int_0^{2\pi}\left(c_0c_1-4\alpha^2c_1-2\alpha c_2\right)\cos\phi d\phi dt_1dt_2 \\
&= 3\left\langle M_1x_1^2\right\rangle+\left\langle M_1x_2^2\right\rangle+2\left\langle M_1x_1x_2\right\rangle+8h\left\langle M_3x_1\right\rangle-4h^2m_1 \qquad (3.25)
\end{aligned}
$$

completes the estimate of I_1:

$$
I_1\simeq\frac{m_1\gamma}{2\rho_0^{3/2}}+\frac{g_1}{2\rho_0^2}-\frac{1}{4\gamma\rho_0^{5/2}}\left[2m_1\gamma^2\left(h^2+\frac{\gamma^2}{8}\right)+3q_1\right]-\frac{g_1h}{2\rho_0^3}.
$$

Starting with estimation of the J_1 part of (3.19), we factor out $\dfrac{\sin\theta}{1+\cos\theta}$ from all the terms in both numerator and denominator and expand the resulting denominator as

$$
\frac{1}{\left[1-\dfrac{2\left(1+\cos\theta\right)}{\rho_0\sin\theta}\left(t_1\cos\phi+t_2\sin\phi\right)+\dfrac{\left(1+\cos\theta\right)^2}{\rho_0^2\sin^2\theta}\left(t_1^2+t_2^2+h^2\right)\right]^{3/2}}
$$

$$
\simeq 1+\frac{3\left(1+\cos\theta\right)}{\rho_0\sin\theta}\left(t_1\cos\phi+t_2\sin\phi\right)+\frac{15\left(1+\cos\theta\right)^2}{2\rho_0^2\sin^2\theta}\left(t_1\cos\phi+t_2\sin\phi\right)^2-\frac{3\left(1+\cos\theta\right)^2}{2\rho_0^2\sin^2\theta}\left(t_1^2+t_2^2+h^2\right).
$$

Taking into account (3.23) as well as following orthogonality relations

$$
\int_0^{2\pi}\alpha^2\cos\phi d\phi=\int_0^{2\pi}\alpha\cos^2\phi d\phi=\int_0^{2\pi}\alpha\sin\phi\cos\phi d\phi=0, \qquad (3.26)
$$

we arrive at

$$
J_1 \simeq \frac{1}{4\sqrt{2}\rho_0}\left(1-\frac{h}{\rho_0}\right)\left[m_1\int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi}\left(1+\cos\theta\right)^{3/2}d\theta+\frac{3}{8\rho_0^2}q_1\int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi}\frac{\left(1+\cos\theta\right)^{7/2}}{\sin^2\theta}d\theta\right].
$$

Using half-angle substitution, the integrals in θ variable can be easily computed

$$
\int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi}\left(1+\cos\theta\right)^{3/2}d\theta=4\sqrt{2}\left(\sin\frac{\theta}{2}-\frac{1}{3}\sin^3\frac{\theta}{2}\right)\Bigg|_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi}\simeq 4\sqrt{2}\left(\frac{2}{3}-\frac{\gamma}{2\sqrt{\rho_0}}+\frac{\gamma^3}{16\rho_0^{3/2}}\right), \qquad (3.27)
$$

$$
\int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi}\frac{\left(1+\cos\theta\right)^{7/2}}{\sin^2\theta}d\theta=-4\sqrt{2}\left(\frac{1}{\sin\frac{\theta}{2}}+2\sin\frac{\theta}{2}-\frac{1}{3}\sin^3\frac{\theta}{2}\right)\Bigg|_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi}\simeq 8\sqrt{2}\left(\frac{1}{\gamma}\sqrt{\rho_0}-\frac{4}{3}\right), \qquad (3.28)
$$

leading to

$$
J_1\simeq\frac{2m_1}{3\rho_0}-\frac{m_1\gamma}{2\rho_0^{3/2}}-\frac{2m_1h}{3\rho_0^2}+\frac{1}{4\gamma\rho_0^{5/2}}\left[2m_1\gamma^2\left(h^2+\frac{\gamma^2}{8}\right)+3q_1\right]-\frac{q_1}{\rho_0^3}.
$$

In the final result, all γ-dependent terms cancel as expected, giving

$$
\begin{aligned}
\left\langle\mathcal{K}\left[\Phi\right],S_1^{-1}\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} &= I_1+J_1=\frac{2m_1}{3\rho_0}+\frac{1}{6\rho_0^2}\left(3g_1-4m_1h\right)-\frac{1}{2\rho_0^3}\left(2q_1+g_1h\right)+\mathcal{O}\left(\frac{1}{\rho_0^4}\right) \qquad (3.29)\\
&= \frac{2m_1}{3R_0}+\frac{1}{6R_0^2}\left(3g_1-8m_1h\right)-\frac{1}{2R_0^3}\left(2q_1+3g_1h-4m_1h^2\right)+\mathcal{O}\left(\frac{1}{R_0^4}\right). \qquad (3.30)
\end{aligned}
$$

In totally analogous fashion, we obtain

$$\left\langle \mathcal{K}\left[\Phi\right], S_1^1\right\rangle_{L^2(\mathbb{S}_{R_0})} = \frac{2m_2}{3\rho_0} + \frac{1}{6\rho_0^2}\left(3g_2 - 4m_2 h\right) - \frac{1}{2\rho_0^3}\left(2q_2 + g_2 h\right) + \mathcal{O}\left(\frac{1}{\rho_0^4}\right) \tag{3.31}$$

$$= \frac{2m_2}{3R_0} + \frac{1}{6R_0^2}\left(3g_2 - 8m_2 h\right) - \frac{1}{2R_0^3}\left(2q_2 + 3g_2 h - 4m_2 h^2\right) + \mathcal{O}\left(\frac{1}{R_0^4}\right) \tag{3.32}$$

with

$$\begin{aligned} q_2 &:= -\frac{1}{\pi}\iint_Q \int_0^{2\pi} \left(c_0 c_1 - 4\alpha^2 c_1 - 2\alpha c_2\right)\sin\phi \, d\phi dt_1 dt_2 \\ &= 3\left\langle M_2 x_2^2\right\rangle + \left\langle M_2 x_1^2\right\rangle + 2\left\langle M_2 x_1 x_2\right\rangle + 8h\left\langle M_3 x_2\right\rangle - 4h^2 m_2, \end{aligned} \tag{3.33}$$

$$g_2 := \frac{1}{\pi}\iint_Q \int_0^{2\pi} c_3 \sin\phi \, d\phi dt_1 dt_2. \tag{3.34}$$

This result confirms the intuition that tangential components of the net moment is contained at the leading order of projection of transformed potential onto the first spherical harmonic: with help of identities (3.18), estimates (3.29)-(3.31) furnish

$$\begin{aligned} m_1 &= \frac{3}{2}\lim_{R_0\to\infty}\left(R_0 + h\right)\left\langle \mathcal{K}\left[\Phi\right], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} \\ &= 3\lim_{R_0\to\infty} R_0^3\left\langle \Phi, \mathcal{K}\left[S_1^{-1}\right]\right\rangle_{L^2_w(\mathbb{R}^2)} = 6\lim_{R_0\to\infty} R_0^4 \iint_{\mathbb{R}^2} \Phi\left(\mathbf{x}, h\right) \frac{x_1}{\left[x_1^2 + x_2^2 + \left(R_0 + h\right)^2\right]^{5/2}} dx_1 dx_2, \end{aligned} \tag{3.35}$$

$$m_2 = 3\lim_{R_0\to\infty} R_0^3\left\langle \Phi, \mathcal{K}\left[S_1^{1}\right]\right\rangle_{L^2_w(\mathbb{R}^2)} = 6\lim_{R_0\to\infty} R_0^4 \iint_{\mathbb{R}^2} \Phi\left(\mathbf{x}, h\right) \frac{x_2}{\left[x_1^2 + x_2^2 + \left(R_0 + h\right)^2\right]^{5/2}} dx_1 dx_2. \tag{3.36}$$

We notice that asymptotic expansions (3.29)-(3.31) contain essentially meaningless information about magnetization at the second order term in $1/\rho_0$ whereas the third order term includes a more valuable piece of information (a combination of higher-order algebraic moments of magnetization). It is at this point where the second Kelvin transform $\tilde{\mathcal{K}}$ comes in handy. As we shall now see, both transforms can be used in conjunction to give a degree of suppressing "magnetic garbage" terms g_1, g_2 defined in (3.24), (3.34) with almost no extra computational effort. We will demonstrate possibility of efficient elimination of "magnetic garbage" terms later on, for now we just give analogs of formulas (3.29)-(3.31) for $\tilde{\mathcal{K}}$.

Denoting $\tilde{\rho}_0 := R_0 - h$, we observe that the expression

$$\begin{aligned} \left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} &= \frac{1 + h/\tilde{\rho}_0}{4\pi\tilde{\rho}_0}\int_0^{\pi}\int_0^{2\pi} \frac{\sin^2\theta\cos\phi}{\sqrt{2\left(1-\cos\theta\right)}} \iint_Q \left[M_1\left(\mathbf{t}\right)\left(\frac{\sin\theta\cos\phi}{1-\cos\theta} - \frac{t_1}{\tilde{\rho}_0}\right)\right. \\ &\left. + M_2\left(\mathbf{t}\right)\left(\frac{\sin\theta\sin\phi}{1-\cos\theta} - \frac{t_2}{\tilde{\rho}_0}\right) + M_3\left(\mathbf{t}\right)\frac{h}{\tilde{\rho}_0}\right] \\ &\times \frac{dt_1 dt_2 d\phi d\theta}{\left[\left(\frac{\sin\theta\cos\phi}{1-\cos\theta} - \frac{t_1}{\tilde{\rho}_0}\right)^2 + \left(\frac{\sin\theta\sin\phi}{1-\cos\theta} - \frac{t_2}{\tilde{\rho}_0}\right)^2 + \frac{h^2}{\tilde{\rho}_0^2}\right]^{3/2}}, \end{aligned} \tag{3.37}$$

after the change of variable $\theta \to \pi - \theta$, differs from its counterpart (3.19) only by inverse sign in front of all instances of h except those in the combination $M_3 h$. This immediately entails

$$\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_1^{-1} \right\rangle_{L^2(\mathbb{S}_{R_0})} = \frac{2m_1}{3\tilde{\rho}_0} + \frac{1}{6\tilde{\rho}_0^2}\left(3g_1 + 4m_1 h\right) - \frac{1}{2\tilde{\rho}_0^3}\left(2q_1 - g_1 h\right) + \mathcal{O}\left(\frac{1}{\tilde{\rho}_0^4}\right) \tag{3.38}$$

$$= \frac{2m_1}{3R_0} + \frac{1}{6R_0^2}\left(3g_1 + 8m_1 h\right) - \frac{1}{2R_0^3}\left(2q_1 - 3g_1 h - 4m_1 h^2\right) + \mathcal{O}\left(\frac{1}{R_0^4}\right), \tag{3.39}$$

and similarly

$$\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_1^{1} \right\rangle_{L^2(\mathbb{S}_{R_0})} = \frac{2m_2}{3\tilde{\rho}_0} + \frac{1}{6\tilde{\rho}_0^2}\left(3g_2 + 4m_2 h\right) - \frac{1}{2\tilde{\rho}_0^3}\left(2q_2 - g_2 h\right) + \mathcal{O}\left(\frac{1}{\tilde{\rho}_0^4}\right) \tag{3.40}$$

$$= \frac{2m_2}{3R_0} + \frac{1}{6R_0^2}\left(3g_2 + 8m_2 h\right) - \frac{1}{2R_0^3}\left(2q_2 - 3g_2 h - 4m_2 h^2\right) + \mathcal{O}\left(\frac{1}{R_0^4}\right). \tag{3.41}$$

Now we are going to lift these results to those of more practical importance, that is to obtain a similar formula involving only the normal component of field rather than potential. In view of presence of the ξ_3 factor in formulas (3.11), (3.15) and the fact that $S_1^{-1}(\theta,\phi)\cos\theta = S_2^{-1}(\theta,\phi)$, $S_1^{1}(\theta,\phi)\cos\theta = S_2^{1}(\theta,\phi)$, we need to evaluate projections of the transformed potential onto second order spherical harmonics

$$\left\langle \mathcal{K}\left[\Phi\right], S_2^{-1} \right\rangle_{L^2(\mathbb{S}_{R_0})} = \frac{2m_1}{5\rho_0} + \frac{1}{10\rho_0^2}\left(5g_1 - 4m_1 h\right) - \frac{1}{10\rho_0^3}\left(14q_1 + 5g_1 h\right) + \mathcal{O}\left(\frac{1}{\rho_0^4}\right), \tag{3.42}$$

$$\left\langle \mathcal{K}\left[\Phi\right], S_2^{1} \right\rangle_{L^2(\mathbb{S}_{R_0})} = \frac{2m_2}{5\rho_0} + \frac{1}{10\rho_0^2}\left(5g_2 - 4m_2 h\right) - \frac{1}{10\rho_0^3}\left(14q_2 + 5g_2 h\right) + \mathcal{O}\left(\frac{1}{\rho_0^4}\right). \tag{3.43}$$

We observe that the presence of $\cos\theta$, due to non-vanishing at $\theta = 0$, does not shift the balance between local and global terms in the asymptotic estimate of the integral. While intermediate γ terms are affected with this modification, these terms still vanish in the final result, and the difference in numerical coefficients of expressions (3.42)-(3.43) compared to (3.29)-(3.31) comes only from constant (γ-independent) terms in expansion of integrals

$$\int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi} \left(1+\cos\theta\right)^{3/2}\cos\theta d\theta = 4\sqrt{2}\left(\sin\frac{\theta}{2} - \sin^3\frac{\theta}{2} + \frac{2}{5}\sin^5\frac{\theta}{2}\right)\Bigg|_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi} = \frac{8}{5}\sqrt{2} + [\gamma \text{ terms}], \tag{3.44}$$

$$\int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi} \frac{\left(1+\cos\theta\right)^{7/2}}{\sin^2\theta}\cos\theta d\theta = -4\sqrt{2}\left(\frac{1}{\sin\frac{\theta}{2}} + 4\sin\frac{\theta}{2} - \frac{5}{3}\sin^3\frac{\theta}{2} + \frac{2}{5}\sin^5\frac{\theta}{2}\right)\Bigg|_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi} = -\frac{224}{15}\sqrt{2} + [\gamma \text{ terms}], \tag{3.45}$$

In case of $\tilde{\mathcal{K}}$, in addition to the formal replacement $h \to -h$ already discussed, there is an overall inversion of sign due to an extra cosine factor which reverses the sign under change of variable $\theta \to \pi - \theta$

$$\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_2^{-1} \right\rangle_{L^2(\mathbb{S}_{R_0})} = -\frac{2m_1}{5\tilde{\rho}_0} - \frac{1}{10\tilde{\rho}_0^2}\left(5g_1 + 4m_1 h\right) + \frac{1}{10\tilde{\rho}_0^3}\left(14q_1 - 5g_1 h\right) + \mathcal{O}\left(\frac{1}{\tilde{\rho}_0^4}\right), \tag{3.46}$$

$$\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_2^{1} \right\rangle_{L^2(\mathbb{S}_{R_0})} = -\frac{2m_2}{5\tilde{\rho}_0} - \frac{1}{10\tilde{\rho}_0^2}\left(5g_2 + 4m_2 h\right) + \frac{1}{10\tilde{\rho}_0^3}\left(14q_2 - 5g_2 h\right) + \mathcal{O}\left(\frac{1}{\tilde{\rho}_0^4}\right). \tag{3.47}$$

Remaining ingredient, namely, terms with radial derivatives of the transformed potential, can be obtained using expansions over solid harmonics of $\mathcal{K}[\Phi]$ and $\tilde{\mathcal{K}}[\Phi]$ in their regions of harmonicity

$$\mathcal{K}[\Phi](r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{m,l} r^l S_l^m(\theta,\phi), \quad r \le R_0, \tag{3.48}$$

$$\tilde{\mathcal{K}}[\Phi](r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \tilde{C}_{m,l} \frac{1}{r^{l+1}} S_l^m(\theta,\phi), \quad r \ge R_0, \tag{3.49}$$

where $C_{m,l}$, $\tilde{C}_{m,l} \in \mathbb{R}$, $l \in \mathbb{Z}_0$, $m = [-l, \ldots, l]$.

Differentiation of these expansions with respect to r and orthogonality of spherical harmonics lead to the following identities

$$\left\langle \partial_r \mathcal{K}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = \frac{1}{R_0}\left\langle \mathcal{K}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})}, \quad \left\langle \partial_r \mathcal{K}[\Phi], S_2^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = \frac{2}{R_0}\left\langle \mathcal{K}[\Phi], S_2^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})},$$

$$\left\langle \partial_r \tilde{\mathcal{K}}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = -\frac{2}{R_0}\left\langle \tilde{\mathcal{K}}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})}, \quad \left\langle \partial_r \tilde{\mathcal{K}}[\Phi], S_2^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = -\frac{3}{R_0}\left\langle \tilde{\mathcal{K}}[\Phi], S_2^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})}.$$

Plugging this into (3.11), (3.15) and employing (3.29), (3.38), (3.42), (3.46) we obtain

$$\begin{aligned}\left\langle \mathcal{K}[\partial_{x_3}\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} &= -\frac{1}{2\rho_0}\left(3\left\langle \mathcal{K}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} + 5\left\langle \mathcal{K}[\Phi], S_2^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})}\right)\\ &= -\frac{2m_1}{R_0^2} - \frac{2}{R_0^3}(g_1 - 3m_1 h) + \frac{1}{R_0^4}\left(5q_1 + 8g_1 h - 12m_1 h^2\right) + \mathcal{O}\left(\frac{1}{R_0^5}\right),\end{aligned}$$

$$\begin{aligned}\left\langle \tilde{\mathcal{K}}[\partial_{x_3}\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} &= -\frac{1}{2\tilde{\rho}_0}\left(3\left\langle \tilde{\mathcal{K}}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} - 5\left\langle \tilde{\mathcal{K}}[\Phi], S_2^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})}\right)\\ &= -\frac{2m_1}{R_0^2} - \frac{2}{R_0^3}(g_1 + 3m_1 h) + \frac{1}{R_0^4}\left(5q_1 - 8g_1 h - 12m_1 h^2\right) + \mathcal{O}\left(\frac{1}{R_0^5}\right).\end{aligned}$$

Similarly,

$$\left\langle \mathcal{K}[\partial_{x_3}\Phi], S_1^{1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = -\frac{2m_2}{R_0^2} - \frac{2}{R_0^3}(g_2 - 3m_2 h) + \frac{1}{R_0^4}\left(5q_2 + 8g_2 h - 12m_2 h^2\right) + \mathcal{O}\left(\frac{1}{R_0^5}\right),$$

$$\left\langle \tilde{\mathcal{K}}[\partial_{x_3}\Phi], S_1^{1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = -\frac{2m_2}{R_0^2} - \frac{2}{R_0^3}(g_2 + 3m_2 h) + \frac{1}{R_0^4}\left(5q_2 - 8g_2 h - 12m_2 h^2\right) + \mathcal{O}\left(\frac{1}{R_0^5}\right).$$

In particular, this leads to the desired formulas for the tangential components of the net moment in terms of field data

$$\begin{aligned} m_1 &= \frac{1}{2}\lim_{R_0\to\infty} R_0^2 \left\langle \mathcal{K}[B_3], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} = \lim_{R_0\to\infty} R_0^4 \left\langle B_3, \mathcal{K}\left[S_1^{-1}\right]\right\rangle_{L^2_w(\mathbb{R}^2)}\\ &= 2\lim_{R_0\to\infty} R_0^5 \iint_{\mathbb{R}^2} B_3(\mathbf{x},h)\,\frac{x_1}{\left[x_1^2 + x_2^2 + (R_0+h)^2\right]^{5/2}} dx_1 dx_2, \end{aligned} \tag{3.50}$$

$$
\begin{aligned}
m_2 &= \frac{1}{2}\lim_{R_0\to\infty} R_0^2 \left\langle \mathcal{K}\left[B_3\right], S_1^1\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = \lim_{R_0\to\infty} R_0^4 \left\langle B_3, \mathcal{K}\left[S_1^1\right]\right\rangle_{L^2_w(\mathbb{R}^2)} \\
&= 2\lim_{R_0\to\infty} R_0^5 \iint_{\mathbb{R}^2} B_3\left(\mathbf{x},h\right) \frac{x_2}{\left[x_1^2+x_2^2+\left(R_0+h\right)^2\right]^{5/2}} dx_1 dx_2.
\end{aligned} \tag{3.51}
$$

3.3.2 Recovery of other algebraic moments

It seems straightforward that considering integration against the spherical harmonic S_1^0 will yield an estimate for the normal component of net moment m_3 as it would be an exact result in case of available data on a sphere rather than plane in physical space. However, we are going to see that the situation is slightly different.

As before, we consider

$$
\begin{aligned}
\left\langle \mathcal{K}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} &= \frac{1-h/\rho_0}{4\pi\rho_0}\int_0^{\pi}\int_0^{2\pi} \frac{\sin\theta\cos\theta}{\sqrt{2\left(1+\cos\theta\right)}} \iint_Q \left[M_1\left(\mathbf{t}\right)\left(\frac{\sin\theta\cos\phi}{1+\cos\theta}-\frac{t_1}{\rho_0}\right)\right. \\
&\left. + M_2\left(\mathbf{t}\right)\left(\frac{\sin\theta\sin\phi}{1+\cos\theta}-\frac{t_2}{\rho_0}\right) + M_3\left(\mathbf{t}\right)\frac{h}{\rho_0}\right] \\
&\times \frac{dt_1 dt_2 d\phi d\theta}{\left[\left(\frac{\sin\theta\cos\phi}{1+\cos\theta}-\frac{t_1}{\rho_0}\right)^2+\left(\frac{\sin\theta\sin\phi}{1+\cos\theta}-\frac{t_2}{\rho_0}\right)^2+\frac{h^2}{\rho_0^2}\right]^{3/2}}.
\end{aligned} \tag{3.52}
$$

The main difference of the results of estimation comes from the fact that the absence of $\sin\theta$ factor results in that the local contribution to the integral from the neighborhood of $\theta = 0$ is now prevalent over the global one which after integration in ϕ becomes of order $\mathcal{O}\left(1/\rho_0^2\right)$ as leading order terms proportional to M_1 and M_2 vanish. We note that it is a global contribution that generally contains meaningful information which, in this case, was expected to give an estimate of m_3, whereas a local contribution produces contamination that we labeled as "magnetic garbage". Unlike in previous computations, we perform here only two-term asymptotic expansion; however, by little effort the expansion can be continued further.

Following precisely the same steps as before, we let

$$
\left\langle \mathcal{K}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = \int_0^{\frac{\gamma}{\sqrt{\rho_0}}} \left(\dots\right) d\theta + \int_{\frac{\gamma}{\sqrt{\rho_0}}}^{\pi} \left(\dots\right) d\theta =: I_0 + J_0.
$$

In addition to previously mentioned computational pieces, we employ the Taylor expansion

$$
\frac{\sin\theta\cos\theta}{\sqrt{2\left(1+\cos\theta\right)}} \simeq \frac{\theta}{2}\left(1-\frac{13}{24}\theta^2\right)
$$

and estimate

$$
\begin{aligned}
I_0 \simeq\ & \frac{1}{2\pi\rho_0^3}\left(1-\frac{h}{\rho_0}\right)\iint_Q \int_0^{2\pi}\left[\rho_0^2\int_0^{\gamma\sqrt{\rho_0}} \frac{\left(c_1\omega+c_2\right)\omega}{\left[\left(\omega-\alpha\right)^2+c_0\right]^{3/2}} d\omega - \frac{13}{24}\int_0^{\gamma\sqrt{\rho_0}} \frac{\left(c_1\omega+c_2\right)\omega^3}{\left[\left(\omega-\alpha\right)^2+c_0\right]^{3/2}} d\omega\right. \\
&\left. -\frac{1}{4}\int_0^{\gamma\sqrt{\rho_0}} \frac{\left(c_1\omega+c_2\right)\left(\omega-\alpha\right)\omega^4}{\left[\left(\omega-\alpha\right)^2+c_0\right]^{5/2}} d\omega + \frac{1}{12}\int_0^{\gamma\sqrt{\rho_0}} \frac{c_1\omega^4}{\left[\left(\omega-\alpha\right)^2+c_0\right]^{3/2}} d\omega\right] d\phi dt_1 dt_2.
\end{aligned} \tag{3.53}
$$

Again, due to the factor ρ_0^2 dictating the necessity of higher-order expansion, we perform explicit integration in the first integral (see Appendix) followed by asymptotical expansion whereas the other terms are estimated asymptotically in direct manner:

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,\omega}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \simeq c_1 \log\left(2\gamma\sqrt{\rho_0}\right) - c_1 + c_4 - \frac{c_2 + 3\alpha c_1}{\gamma\sqrt{\rho_0}},$$

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,\omega^3}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \simeq \frac{1}{2}c_1\gamma^2\rho_0 + (3\alpha c_1 + c_2)\,\gamma\sqrt{\rho_0},$$

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{(c_1\omega + c_2)\,(\omega-\alpha)\,\omega^4}{\left[(\omega-\alpha)^2 + c_0\right]^{5/2}} d\omega \simeq \frac{1}{2}c_1\gamma^2\rho_0 + (4\alpha c_1 + c_2)\,\gamma\sqrt{\rho_0},$$

$$\int_0^{\gamma\sqrt{\rho_0}} \frac{c_1\omega^4}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \simeq \frac{1}{2}c_1\gamma^2\rho_0 + 3\alpha c_1\gamma\sqrt{\rho_0},$$

where

$$c_4 := -\frac{1}{2}c_1 \log c_0 + c_1 \operatorname{arcsinh}\frac{\alpha}{\sqrt{c_0}} + \frac{\alpha}{c_0}\left(\alpha c_1 + c_2\right) + \frac{\alpha\left(-c_0c_1 + \alpha\left(\alpha c_1 + c_2\right)\right) + 2\alpha c_1 + c_2}{c_0\left(\alpha^2 + c_0\right)^{1/2}}.$$

Since

$$\int_0^{2\pi} c_1 d\phi = 0, \tag{3.54}$$

we have vanishing of leading order terms in all the integrals above, at the same time the terms of order $\mathcal{O}\left(\rho_0^{1/2}\right)$ will be already out of the scope of interest and their cancelation should not even be traced. We thus end up with

$$I_0 \simeq \frac{g_0}{2\rho_0} - \frac{q_0}{\gamma\rho_0^{3/2}} - \frac{g_0 h}{2\rho_0^2},$$

where

$$g_0 := \frac{1}{\pi}\iint_Q \int_0^{2\pi} c_4 d\phi dt_1 dt_2, \tag{3.55}$$

$$q_0 := \frac{1}{2\pi}\iint_Q \int_0^{2\pi} (3\alpha c_1 + c_2)\, d\phi dt_1 dt_2 = \langle M_1 x_1\rangle + \langle M_2 x_2\rangle + 2hm_3. \tag{3.56}$$

Due to (3.54), computation of the J_0 part essentially becomes

$$\begin{aligned} J_0 &\simeq \frac{q_0}{4\rho_0}\frac{1}{\sqrt{2}}\int_{\gamma/\sqrt{\rho_0}}^{\pi} \frac{\cos\theta\,(1+\cos\theta)^{5/2}}{\sin^2\theta} d\theta \simeq -\frac{2q_0}{\rho_0}\left(\frac{1}{\sin\dfrac{\theta}{2}} + 3\sin\frac{\theta}{2} - \frac{2}{3}\sin^3\frac{\theta}{2}\right)\Bigg|_{\gamma/\sqrt{\rho_0}}^{\pi} \\ &\simeq -\frac{5q_0}{3\rho_0^2} + \frac{q_0}{\gamma\rho_0^{3/2}}. \end{aligned}$$

Finally,

$$\begin{aligned}\left\langle \mathcal{K}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} &= \frac{g_0}{2\rho_0} - \frac{5q_0}{3\rho_0^2} + \mathcal{O}\left(\frac{1}{\rho_0^3}\right) \\ &= \frac{g_0}{2R_0} - \frac{1}{3R_0^2}\left(5q_0 + 3g_0 h\right) + \mathcal{O}\left(\frac{1}{R_0^3}\right).\end{aligned} \tag{3.57}$$

By the change of variable trick described before, we immediately obtain a counterpart of this formula

$$\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = -\frac{g_0}{2R_0} + \frac{1}{3R_0^2}\left(5q_0 - 3g_0 h\right) + \mathcal{O}\left(\frac{1}{R_0^3}\right). \tag{3.58}$$

It is at this point that we can appreciate a combination of two transforms. The difference of the two expressions above neatly allows us to filter the meaningful quantity q_0 from the contaminating term g_0 at the order $\mathcal{O}\left(1/R_0^2\right)$

$$\left\langle \left(\mathcal{K} - \tilde{\mathcal{K}}\right)\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = \frac{g_0}{R_0} - \frac{10q_0}{3R_0^2} + \mathcal{O}\left(\frac{1}{R_0^3}\right).$$

It only remains to suppress the presence of the leading order term. This can be done, for example, by combining two instances of this expression evaluated at two different values of R_0 which are large enough for validity of the asymptotic expansions. For instance,

$$\begin{aligned} q_0 &= \frac{3\varrho^2}{5}\left[4\left\langle \left(\mathcal{K} - \tilde{\mathcal{K}}\right)\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)}\Big|_{R_0=2\varrho} - \left\langle \left(\mathcal{K} - \tilde{\mathcal{K}}\right)\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)}\Big|_{R_0=\varrho}\right] + \mathcal{O}\left(\frac{1}{\varrho}\right) \\ &= \frac{6}{5}\lim_{\varrho\to\infty}\varrho^3 \iint_{\mathbb{R}^2} B_3\left(\mathbf{x}, h\right)\left[\frac{4\left(2\varrho+h\right)\left[\left(2\varrho+h\right)^2 - \varrho^2\right]}{\left[x_1^2 + x_2^2 + \left(2\varrho+h\right)^2\right]^{5/2}} + \frac{4\left(2\varrho-h\right)\left[\left(2\varrho+h\right)^2 - \varrho^2\right]}{\left[x_1^2 + x_2^2 + \left(2\varrho-h\right)^2\right]^{5/2}}\right. \\ &\quad \left. - \frac{\left(\varrho+h\right)\left[\left(\varrho+h\right)^2 - \varrho^2\right]}{\left[x_1^2 + x_2^2 + \left(\varrho+h\right)^2\right]^{5/2}} - \frac{\left(\varrho-h\right)\left[\left(\varrho-h\right)^2 - \varrho^2\right]}{\left[x_1^2 + x_2^2 + \left(\varrho-h\right)^2\right]^{5/2}}\right] dx_1 dx_2. \end{aligned} \tag{3.59}$$

In order to derive analogous expressions in terms of transformed field, we would need to compute integrals against $S_2^0\left(\theta,\phi\right) = \cos^2\theta$. While it can be done, we instead perform integration simply against the constant $S_0\left(\theta,\phi\right) = 1$ which is the zeroth spherical harmonic and also differs from what has been computed by cosine factor, and hence suitable for the use of formulas (3.11), (3.15).

We note that

$$\int_{\gamma/\sqrt{\rho_0}}^{\pi} \frac{\left(1+\cos\theta\right)^{5/2}}{\sin^2\theta} d\theta = -2\sqrt{2}\left.\frac{1+\sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right|_{\gamma/\sqrt{\rho_0}}^{\pi} \simeq -4\sqrt{2} + \left[\gamma \text{ terms}\right],$$

and this produces essentially the only change in the resulting formulas

$$\left\langle \mathcal{K}\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = \frac{g_0}{2R_0} - \frac{1}{R_0^2}\left(q_0 + 3g_0 h\right) + \mathcal{O}\left(\frac{1}{R_0^3}\right), \tag{3.60}$$

$$\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = \frac{g_0}{2R_0} - \frac{1}{R_0^2}\left(q_0 - 3g_0 h\right) + \mathcal{O}\left(\frac{1}{R_0^3}\right). \tag{3.61}$$

As before, we observe that expansions $\mathcal{K}\left[\Phi\right]$ and $\tilde{\mathcal{K}}\left[\Phi\right]$ over solid harmonics (3.48)-(3.49) and orthogonality of spherical harmonics imply that

$$\left\langle \partial_r \mathcal{K}\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = 0, \quad \left\langle \partial_r \mathcal{K}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = \frac{1}{R_0}\left\langle \mathcal{K}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)},$$

$$\left\langle \partial_r \tilde{\mathcal{K}}\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = -\frac{1}{R_0}\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)}, \quad \left\langle \partial_r \tilde{\mathcal{K}}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = -\frac{2}{R_0}\left\langle \tilde{\mathcal{K}}\left[\Phi\right], S_1^0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)}.$$

Finally, combined with these relations and expansions (3.57)-(3.61), the formulas (3.11), (3.15) give

$$\left\langle \mathcal{K}\left[\partial_{x_3}\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = -\frac{g_0}{R_0^2} + \frac{3}{R_0^3}\left(q_0 + g_0 h\right) + \mathcal{O}\left(\frac{1}{R_0^4}\right),$$

$$\left\langle \tilde{\mathcal{K}}\left[\partial_{x_3}\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = -\frac{g_0}{R_0^2} + \frac{3}{R_0^3}\left(q_0 - g_0 h\right) + \mathcal{O}\left(\frac{1}{R_0^4}\right).$$

We sum the expressions aiming to isolate q_0 from the "magnetic garbage" g_0 at order $\mathcal{O}\left(1/R_0^3\right)$

$$\frac{1}{2}\left\langle \left(\mathcal{K} + \tilde{\mathcal{K}}\right)\left[\partial_{x_3}\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)} = -\frac{g_0}{R_0^2} + \frac{3q_0}{R_0^3} + \mathcal{O}\left(\frac{1}{R_0^4}\right),$$

and eliminate the leading order term the same way as before to obtain

$$\begin{aligned} q_0 &= -\frac{\varrho^3}{3}\left[4\left\langle \left(\mathcal{K} + \tilde{\mathcal{K}}\right)\left[\partial_{x_3}\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)}\Big|_{R_0 = 2\varrho} - \left\langle \left(\mathcal{K} + \tilde{\mathcal{K}}\right)\left[\partial_{x_3}\Phi\right], S_0\right\rangle_{L^2\left(\mathbb{S}_{R_0}\right)}\Big|_{R_0 = \varrho}\right] + \mathcal{O}\left(\frac{1}{\varrho}\right) \\ &= \frac{2}{3}\lim_{\varrho\to\infty}\varrho^4 \iint_{\mathbb{R}^2} B_3\left(\mathbf{x}, h\right)\left[\frac{8\left(2\varrho - h\right)}{\left[x_1^2 + x_2^2 + \left(2\varrho - h\right)^2\right]^{3/2}} + \frac{8\left(2\varrho + h\right)}{\left[x_1^2 + x_2^2 + \left(2\varrho + h\right)^2\right]^{3/2}}\right. \\ &\quad \left. - \frac{\varrho + h}{\left[x_1^2 + x_2^2 + \left(\varrho + h\right)^2\right]^{3/2}} - \frac{\varrho - h}{\left[x_1^2 + x_2^2 + \left(\varrho - h\right)^2\right]^{3/2}}\right] dx_1 dx_2. \end{aligned} \tag{3.62}$$

3.4 Normal component of the net moment

As we have seen, the described approach yields only the quantity q_0 defined in (3.56) which seems to be the closest we can get to the recovery of m_3 from the field data, but only if we assume smallness of higher-order algebraic moments $\langle M_3 x_1\rangle$, $\langle M_3 x_2\rangle$. However, in the idealistic case of completely available data, we can do better without additional assumptions if we adopt another technique based on the Poisson representation formula [2] and asymptotic dominance of normal magnetization component in (3.8).

By construction, $\mathcal{K}\left[B_3\right]$ is a function harmonic inside the ball $\mathbb{B}_{R_0}$, and hence it admits representation by means of the Poisson integral formula

$$\mathcal{K}\left[B_3\right](\boldsymbol{\eta}) = \frac{R_0^2 - |\boldsymbol{\eta}|^2}{4\pi R_0} \iint_{\mathbb{S}_{R_0}} \frac{\mathcal{K}\left[B_3\right](\boldsymbol{\xi})}{|\boldsymbol{\xi} - \boldsymbol{\eta}|^3} d\sigma_\xi, \quad \boldsymbol{\eta} \in \mathbb{B}_{R_0},$$

where $d\sigma_\xi = R_0^2 \sin\theta d\theta d\phi$, $\theta = \arctan \dfrac{\sqrt{\xi_1^2 + \xi_2^2}}{\xi_3}$, $\phi = \arctan \dfrac{\xi_2}{\xi_1}$ is a non-normalized Lebesgue measure on the sphere $\mathbb{S}_{R_0}$.

This representation significantly simplifies when restricted to the vertical axis $\eta_1 = \eta_2 = 0$, $-R_0 < \eta_3 < R_0$:

$$\mathcal{K}\left[B_3\right](0, 0, \eta_3) = \frac{R_0^2 - \eta_3^2}{4\pi} \int_{-R_0}^{R_0} \frac{\Lambda^\star(\xi_3)}{\left(R_0^2 - 2\xi_3\eta_3 + \eta_3^2\right)^{3/2}} d\xi_3, \tag{3.63}$$

where $\Lambda^\star(\xi_3) := \int_0^{2\pi} \mathcal{K}\left[B_3\right](\boldsymbol{\xi})\, d\phi$. Moreover, employing (3.13), we can express

$$\Lambda^\star(\xi_3) = \mathcal{K}\left[\int_0^{2\pi} B_3(\rho\cos\varphi, \rho\sin\varphi, h)\, d\varphi\right](\xi_3) = \mathcal{K}\left[\Lambda\right](\boldsymbol{\xi}), \tag{3.64}$$

where

$$\Lambda(\rho) := \int_0^{2\pi} B_3(\rho\cos\varphi, \rho\sin\varphi, h)\, d\varphi \tag{3.65}$$

is essentially the angular average of the measured field.

On the other hand, since Kelvin transformation is a local operation, from (3.8) relaxing $x_3 = h$, we have

$$B_3(\mathbf{0}, x_3) \simeq \frac{m_3}{2\pi x_3^3} \quad \text{as } x_3 \to \infty \quad \Rightarrow \quad \mathcal{K}\left[B_3\right](0, 0, \eta_3) \simeq \frac{m_3}{2\pi} \frac{\left(R_0 + \eta_3\right)^2}{R_0^3\left(R_0 + 2h - \eta_3\right)^3} \quad \text{as } \eta_3 \searrow -R_0,$$

and hence (3.63) implies

$$\frac{2m_3\left(R_0 + \eta_3\right)}{R_0^3\left(R_0 - \eta_3\right)\left(R_0 + 2h - \eta_3\right)^3} + \mathcal{O}\left(R_0 + \eta_3\right)^2 = \int_{-R_0}^{R_0} \frac{\Lambda^\star(\xi_3)}{\left(R_0^2 - 2\xi_3\eta_3 + \eta_3^2\right)^{3/2}} d\xi_3. \tag{3.66}$$

It is remarkable that while vanishing of the left-hand side is immediate, it is not obvious at first glance that

$$\lim_{\eta_3 \to -R_0^+} \int_{-R_0}^{R_0} \frac{\Lambda^\star(\xi_3)}{\left(R_0^2 - 2\xi_3\eta_3 + \eta_3^2\right)^{3/2}} d\xi_3 = 0. \tag{3.67}$$

This last identity is worth discussing since its validity is subtle and hinges on the fact that $\Lambda^\star$ is constructed from B_3 which is a vertical gradient of harmonic in upper-half plane function vanishing at infinity. In fact, it is a consequence of Gauss theorem in disguise. Indeed, integrating div $\mathbf{B} = 0$ above the plane $x_3 = h$ with integration surface closed at infinity, and additionally taking into account that $dx_1 dx_2 = \dfrac{R_0^2}{\left(R_0 + \xi_3\right)^2} d\sigma_\xi$, we have

$$\iint_{\mathbb{R}^2} B_3(\mathbf{x}, h)\, dx_1 dx_2 = 0 \quad \Rightarrow \quad \iint_{\mathbb{S}_{R_0}} \frac{\mathcal{K}\left[B_3\right](\boldsymbol{\xi})}{\left(R_0 + \xi_3\right)^{3/2}} d\sigma_\xi = 0. \tag{3.68}$$

Using (3.64), this last equality will imply (3.67) once we can perform the limit passage.

To this effect, we first note that

$$\Lambda^{\star}(\xi_3) = (R_0 + \xi_3)\,\Lambda_0^{\star}(\xi_3) \tag{3.69}$$

for some $\Lambda_0^{\star} \in C([-1,1])$ which is due to the asymptote $B_3(\mathbf{x}, h) = \mathcal{O}\left(1/|\mathbf{x}|^3\right)$ as $|\mathbf{x}| \to \infty$ and $\mathcal{R}\left[\sqrt{x_1^2 + x_2^2}\right](\boldsymbol{\xi})\Big|_{\xi \in \mathbb{S}_{R_0}} = (R_0 + h)\sqrt{\dfrac{R_0 - \xi_3}{R_0 + \xi_3}}$.

Now, for $-R_0 < \xi_3 < 0$, the inequality

$$R_0^2 - 2\xi_3\eta_3 + \eta_3^2 = (\eta_3 - \xi_3)^2 + R_0^2 - \xi_3^2 \geq (R_0 + \xi_3)(R_0 - \xi_3),$$

allows bounding the integrand in (3.67) by a L^1_{loc} function at $\eta_3 = -R_0$

$$\left| \int_{-R_0}^{0} \frac{\Lambda^{\star}(\xi_3)}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{3/2}} d\xi_3 \right| \leq \int_{-R_0}^{0} \frac{|\Lambda_0^{\star}(\xi_3)|}{(R_0 + \xi_3)^{1/2}(R_0 - \xi_3)^{3/2}} d\xi_3 < \infty,$$

and thus makes dominated convergence theorem [8] applicable to justify the passage to the limit giving

$$\lim_{\eta_3 \to -R_0^+} \int_{-R_0}^{R_0} \frac{\Lambda^{\star}(\xi_3)}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{3/2}} d\xi_3 = \frac{1}{(2R_0)^{3/2}} \int_{-R_0}^{R_0} \frac{\Lambda^{\star}(\xi_3)}{(R_0 + \xi_3)^{3/2}} d\xi_3 = 0$$

with the last equality provided by (3.68).

Now we get back to (3.66) and apply $\left.\dfrac{d}{d\eta_3}\right|_{\eta_3 = -R_0}$ to the both sides of it yielding

$$m_3 = -24R_0^4 (R_0 + h)^3 \lim_{\eta_3 \to -R_0} \int_{-R_0}^{R_0} \frac{\Lambda^{\star}(\xi_3)(\xi_3 - \eta_3)}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{5/2}} d\xi_3.$$

Taking into account that

$$\frac{1}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{5/2}} = \frac{1}{3\eta_3} \frac{d}{d\xi_3} \frac{1}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{3/2}},$$

we perform integration by parts to get

$$m_3 \;=\; 8R_0^4 (R_0 + h)^3 \lim_{\eta_3 \to -R_0} \left[\int_{-R_0}^{R_0} \frac{-\eta_3 \Lambda^{\star}(\xi_3) + (\Lambda^{\star}(\xi_3))'(\xi_3 - \eta_3)}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{3/2}} d\xi_3 - \left. \frac{\Lambda^{\star}(\xi_3)(\xi_3 - \eta_3)}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{3/2}} \right|_{\xi_3 = -R_0}^{\xi_3 = R_0} \right].$$

We note vanishing of the boundary term for $\xi_3 = -R_0$ because of (3.69) whereas the integral term simplifies due to (3.67). Therefore,

$$m_3 = -2R_0 (R_0 + h)\Lambda^{\star}(R_0) + 8R_0^4 (R_0 + h)^3 \lim_{\eta_3 \to -R_0} \int_{-R_0}^{R_0} \frac{(\Lambda^{\star}(\xi_3))'(\xi_3 - \eta_3)}{(R_0^2 - 2\xi_3\eta_3 + \eta_3^2)^{3/2}} d\xi_3. \tag{3.70}$$

The last term has a singularity that still does not admit passage to the limit and hence has to be integrated by parts again. Then, after passing to the limit, we can simplify the result by performing integration by parts back

$$
\begin{aligned}
\lim_{\eta_3\to -R_0}\int_{-R_0}^{R_0}\frac{\left(\Lambda^\star(\xi_3)\right)'(\xi_3-\eta_3)}{\left(R_0^2-2\xi_3\eta_3+\eta_3^2\right)^{3/2}}d\xi_3 &= -\frac{1}{R_0}\left[(\Lambda^\star)'(R_0)+(\Lambda^\star)'(-R_0)\right]-\lim_{\eta_3\to -R_0}\int_{-R_0}^{R_0}\frac{\left[\left(\Lambda^\star(\xi_3)\right)'(\xi_3-\eta_3)\right]'}{\eta_3\left(R_0^2-2\xi_3\eta_3+\eta_3^2\right)^{1/2}}d\xi_3\\
&= -\frac{1}{R_0}\left[(\Lambda^\star)'(R_0)+(\Lambda^\star)'(-R_0)\right]+\frac{1}{R_0\sqrt{2R_0}}\int_{-R_0}^{R_0}\frac{\left[\left(\Lambda^\star(\xi_3)\right)'(\xi_3+R_0)\right]'}{(\xi_3+R_0)^{1/2}}d\xi_3\\
&= -\frac{1}{R_0}\left[(\Lambda^\star)'(R_0)+(\Lambda^\star)'(-R_0)\right]+\frac{1}{R_0\sqrt{2R_0}}\left[(\Lambda^\star)'(R_0)\sqrt{2R_0}\right.\\
&\quad\left.+\frac{1}{2\sqrt{2R_0}}\Lambda^\star(R_0)+\frac{1}{4}\int_{-R_0}^{R_0}\frac{\Lambda^\star(\xi_3)}{(\xi_3+R_0)^{3/2}}d\xi_3\right]\\
&= -\frac{1}{R_0}(\Lambda^\star)'(-R_0)+\frac{1}{4R_0^2}\Lambda^\star(R_0),
\end{aligned}
$$

where in the last equality we have used (3.67), (3.69).

Then, (3.70) simply becomes

$$
m_3=-8R_0^2(R_0+h)^3(\Lambda^\star)'(-R_0)=-8R_0^2(R_0+h)^3\Lambda_0^\star(-R_0),
$$

and finally, recalling (3.64)-(3.65),

$$
\begin{aligned}
m_3 &= -4\sqrt{2}R_0^{3/2}(R_0+h)^3\lim_{\xi_3\to -R_0^+}\frac{1}{(R_0+\xi_3)^{3/2}}\Lambda\left((R_0+h)\sqrt{\frac{R_0-\xi_3}{R_0+\xi_3}}\right) && (3.71)\\
&= -2\lim_{\rho\to\infty}\rho^3\int_0^{2\pi}B_3(\rho\cos\varphi,\rho\sin\varphi,h)\,d\varphi, && (3.72)
\end{aligned}
$$

where we have changed the limiting variable to $\rho:=(R_0+h)\sqrt{\frac{R_0-\xi_3}{R_0+\xi_3}}$.

The obtained expression looks striking because it does not incorporate data on the whole measurement plane, instead it involves only the integrals of $B_3(\mathbf{x},h)$ over circles of infinitely large radii. Even though this solves the problem in the ideal case of complete data, in practice those distant circles are exactly where we necessarilly lack measurements.

3.5 The case of incomplete data

Now we move on to the more realistic case when the data are measured within an area of finite size rather than the whole plane. We assume this measurement area is the disk D_A of radius A centered above the origin, and available data are the field measurements there. If one sets $B_3(\mathbf{x},h)\equiv 0$ outside of D_A and attempts to reconstruct the net moment using formulas (3.50)-(3.51) for large values of R_0, the resulting estimates will look quite disappointing unless A is an extremely large number which is typically impossible due to practical restrictions. Instead, we extend the measurements by asymptotic expansion of the field at infinity that can be performed directly from

(3.8) using the following Taylor expansions

$$\begin{aligned}\frac{1}{\left[\left(x_1-t_1\right)^2+\left(x_2-t_2\right)^2+x_3^2\right]^{3/2}} &= \frac{1}{\left(x_1^2+x_2^2\right)^{3/2}}\left(1-2\frac{x_1t_1+x_2t_2}{x_1^2+x_2^2}+\frac{t_1^2+t_2^2+h^2}{x_1^2+x_2^2}\right)^{-3/2} \\ &\simeq \frac{1}{\left(x_1^2+x_2^2\right)^{3/2}}\left(1+3\frac{x_1t_1+x_2t_2}{x_1^2+x_2^2}-\frac{3}{2}\frac{t_1^2+t_2^2+h^2}{x_1^2+x_2^2}+\frac{15}{2}\frac{\left(x_1t_1+x_2t_2\right)^2}{\left(x_1^2+x_2^2\right)^2}\right),\end{aligned}$$

and

$$\frac{1}{\left[\left(x_1-t_1\right)^2+\left(x_2-t_2\right)^2+x_3^2\right]^{5/2}} \simeq \frac{1}{\left(x_1^2+x_2^2\right)^{5/2}}\left(1+5\frac{x_1t_1+x_2t_2}{x_1^2+x_2^2}-\frac{5}{2}\frac{t_1^2+t_2^2+h^2}{x_1^2+x_2^2}\right),$$

giving, for $|\mathbf{x}| \gg d_Q+h$,

$$\begin{aligned}B_3\left(\mathbf{x},h\right) \simeq\ & \frac{3h}{4\pi\left(x_1^2+x_2^2\right)^{5/2}}\Bigg[m_1x_1+m_2x_2-\langle M_1x_1\rangle-\langle M_2x_2\rangle+m_3h \\ & +5\left(\langle M_1x_1\rangle\frac{x_1^2}{x_1^2+x_2^2}+\left(\langle M_1x_2\rangle+\langle M_2x_1\rangle\right)\frac{x_1x_2}{x_1^2+x_2^2}+\langle M_2x_2\rangle\frac{x_2^2}{x_1^2+x_2^2}\right)\Bigg] \\ & -\frac{1}{4\pi\left(x_1^2+x_2^2\right)^{3/2}}\Bigg[m_3+3\langle M_3x_1\rangle\frac{x_1}{x_1^2+x_2^2}+3\langle M_3x_2\rangle\frac{x_2}{x_1^2+x_2^2}-\frac{3}{2}\frac{\langle M_3x_1^2\rangle+\langle M_3x_2^2\rangle+m_3h^2}{x_1^2+x_2^2} \\ & +\frac{15}{2}\left(\langle M_3x_1^2\rangle\frac{x_1^2}{\left(x_1^2+x_2^2\right)^2}+2\langle M_3x_1x_2\rangle\frac{x_1x_2}{\left(x_1^2+x_2^2\right)^2}+\langle M_3x_2^2\rangle\frac{x_2^2}{\left(x_1^2+x_2^2\right)^2}\right)\Bigg] \qquad (3.73)\end{aligned}$$

with the next-order terms being proportional to $\dfrac{x_1}{\left(x_1^2+x_2^2\right)^{7/2}}, \dfrac{x_2}{\left(x_1^2+x_2^2\right)^{7/2}}, \dfrac{1}{\left(x_1^2+x_2^2\right)^{7/2}}$.

We decompose the integral in (3.50) into two parts

$$m_1 = \lim_{R_0\to\infty}\iint_{\mathbb{R}^2} B_3\left(\mathbf{x},h\right)\frac{2R_0^5x_1}{\left(x_1^2+x_2^2+\left(R_0+h\right)^2\right)^{5/2}}dx_1dx_2 = \iint_{D_A}\cdots+\iint_{\mathbb{R}^2\backslash D_A}\cdots =: \mathcal{I}_1+\mathcal{J}_1, \quad (3.74)$$

and observe that the dominated convergence theorem can be applied to allow passing to the limit as $R_0\to\infty$ in the first term, while the second term can be computed using asymptotic expansion (3.73) which simplifies by symmetry of the integration area $\mathbb{R}^2\backslash D_A$. That is,

$$\mathcal{I}_1 = 2\iint_{D_A} B_3\left(\mathbf{x},h\right)x_1dx_1dx_2,$$

$$\begin{aligned}\mathcal{J}_1 &\simeq \frac{3}{2\pi}\left(m_1h-\langle M_3x_1\rangle\right)\lim_{R_0\to\infty}R_0^5\iint_{\mathbb{R}^2\backslash D_A}\frac{x_1^2}{\left(x_1^2+x_2^2\right)^{5/2}\left(x_1^2+x_2^2+\left(R_0+h\right)^2\right)^{5/2}}dx_1dx_2 \\ &= \frac{3}{2}\left(m_1h-\langle M_3x_1\rangle\right)\lim_{R_0\to\infty}R_0^5\int_A^\infty\frac{1}{r^2\left(r^2+\left(R_0+h\right)^2\right)^{5/2}}dr = \frac{3}{2A}\left(m_1h-\langle M_3x_1\rangle\right),\end{aligned}$$

where calculations for the last integral is performed in Appendix. The approximate equality sign here means that the next-order term in the expansion in powers of $1/A$ is neglected. Indeed,

$$\frac{15}{8\pi}\left(2h\left\langle M_1x_1\right\rangle-\left\langle M_3x_1^2\right\rangle\right)\lim_{R_0\to\infty}R_0^5\iint_{\mathbb{R}^2\setminus D_A}\frac{x_1^2}{\left(x_1^2+x_2^2\right)^{7/2}\left(x_1^2+x_2^2+\left(R_0+h\right)^2\right)^{5/2}}dx_1dx_2$$

$$\propto\lim_{R_0\to\infty}R_0^5\int_A^\infty\frac{dr}{r^4\left(r^2+\left(R_0+h\right)^2\right)^{5/2}}=\mathcal{O}\left(\frac{1}{A^3}\right),$$

where the final estimate is due to explicit integral computation given in Appendix.

Therefore, from (3.74), we have the balance relation

$$m_1=2\iint_{D_A}B_3\left(\mathbf{x},h\right)x_1dx_1dx_2+\frac{3}{2A}\left(m_1h-\left\langle M_3x_1\right\rangle\right)+\mathcal{O}\left(\frac{1}{A^3}\right), \tag{3.75}$$

which, in case $\left|\left\langle M_3x_1\right\rangle\right|\ll\left|m_1\right|h$, allow solving for

$$m_1=\frac{4A}{2A-3h}\iint_{D_A}B_3\left(\mathbf{x},h\right)x_1dx_1dx_2+\mathcal{O}\left(\frac{1}{A^3}\right). \tag{3.76}$$

Similarly, assuming $\left|\left\langle M_3x_2\right\rangle\right|\ll\left|m_2\right|h$,

$$m_2=\frac{4A}{2A-3h}\iint_{D_A}B_3\left(\mathbf{x},h\right)x_2dx_1dx_2+\mathcal{O}\left(\frac{1}{A^3}\right). \tag{3.77}$$

Smallness of higher algebraic moments is an assumption that can be justified if magnetization decays towards edges of the sample whereas the resulting formulas in this case provide better estimate for the net moment components in terms of available data.

Analogous assumptions could be made on $\left\langle M_3x_1\right\rangle$, $\left\langle M_3x_2\right\rangle$ to isolate m_3 in (3.56) in order to have the expression for the normal component of the net moment in terms of complete data. However, it is remarkable that we still can obtain an expression for the normal component of the net moment m_3 without such assumptions. What is even more surprising is that it can be directly done by means of the same procedure of asymptotic field extension.

Indeed, recalling (3.68), we have

$$0=\iint_{\mathbb{R}^2}B_3\left(\mathbf{x},h\right)dx_1dx_2=\iint_{D_A}B_3\left(\mathbf{x},h\right)dx_1dx_2+\iint_{\mathbb{R}^2\setminus D_A}B_3\left(\mathbf{x},h\right)dx_1dx_2.$$

We compute the second integral on the right the same way as before passing to polar coordinates. The leading order term in (3.73) non-vanishing after the integration is the one proportional to m_3. This immediately gives

$$m_3=2A\iint_{D_A}B_3\left(\mathbf{x},h\right)dx_1dx_2+\mathcal{O}\left(\frac{1}{A^2}\right), \tag{3.78}$$

where the estimate of the neglected term is elementary due to the integral

$$\iint_{\mathbb{R}^2\setminus D_A}\frac{1}{\left(x_1^2+x_2^2\right)^{5/2}}dx_1dx_2=2\pi\int_A^\infty\frac{dr}{r^4}=\frac{2\pi}{3A^3}.$$

The formula (3.78) along with rewritten versions of (3.75)

$$m_1 = 2\iint_{D_A} B_3(\mathbf{x}, h)\, x_1 dx_1 dx_2 + \mathcal{O}\left(\frac{1}{A}\right), \tag{3.79}$$

$$m_2 = 2\iint_{D_A} B_3(\mathbf{x}, h)\, x_2 dx_1 dx_2 + \mathcal{O}\left(\frac{1}{A}\right), \tag{3.80}$$

constitute the most general results of this section.

3.6 Fourier analysis

In this section we are going to see how the formulas for net moments and their generalizations systematically arise from another method based on elementary Fourier analysis.

We rewrite (3.8) in a form

$$B_3(\mathbf{x}, h) = -\frac{1}{4\pi}\iint_Q \left[h\left(M_1(\mathbf{t})\frac{\partial}{\partial x_1}\bigg|_{x_3=h} + M_2(\mathbf{t})\frac{\partial}{\partial x_2}\bigg|_{x_3=h}\right) + M_3(\mathbf{t})\frac{\partial}{\partial x_3}\bigg|_{x_3=h} x_3\right]\left(|\mathbf{x}-\mathbf{t}|^2 + x_3^2\right)^{-3/2} dt_1 dt_2, \tag{3.81}$$

which is convenient for computation of Fourier transform[4]. This yields

$$\hat{B}_3(\mathbf{k}, h) = \pi e^{-2\pi h|\mathbf{k}|}\left[ik_1\hat{M}_1(\mathbf{k}) + ik_2\hat{M}_2(\mathbf{k}) + |\mathbf{k}|\,\hat{M}_3(\mathbf{k})\right]. \tag{3.82}$$

We note that magnetization distribution $\mathbf{M}(\mathbf{t})$ has a compact support and so, by Paley-Wiener theorem [8], its Fourier transform is an entire function. In particular, performing expansion about $\mathbf{k} = \mathbf{0}$, we can extract information about the net moment and higher algebraic moments of magnetization[5]. For $j \in \{1, 2, 3\}$,

$$\begin{aligned}
\hat{M}_j(\mathbf{k}) &= \hat{M}_j(\mathbf{0}) + \partial_{k_1}\hat{M}_j(\mathbf{0})\,k_1 + \partial_{k_2}\hat{M}_j(\mathbf{0})\,k_2 + \frac{1}{2}\left(\partial^2_{k_1}\hat{M}_j(\mathbf{0})\,k_1^2 + \partial^2_{k_2}\hat{M}_j(\mathbf{0})\,k_2^2\right) + \partial_{k_1}\partial_{k_2}\hat{M}_j(\mathbf{0})\,k_1k_2 + \mathcal{O}\left(|\mathbf{k}|^3\right)\\
&= m_j + 2\pi i\left(\langle M_j x_1\rangle k_1 + \langle M_j x_2\rangle k_2\right) - 2\pi^2\left(\langle M_j x_1^2\rangle k_1^2 + \langle M_j x_2^2\rangle k_2^2 + 2\langle M_j x_1 x_2\rangle k_1k_2\right) + \mathcal{O}\left(|\mathbf{k}|^3\right).
\end{aligned}$$

Expanding also the exponential factor,

$$e^{-2\pi h|\mathbf{k}|} = 1 - 2\pi h|\mathbf{k}| + 2\pi^2 h^2|\mathbf{k}|^2 + \mathcal{O}\left(|\mathbf{k}|^3\right),$$

we compute

$$\operatorname{Im}\hat{B}_3(k_1, 0, h) = \pi m_1 k_1 + 2\pi^2\left(\langle M_3 x_1\rangle - m_1 h\right)k_1|k_1| - 2\pi^3\left(\langle M_1 x_1^2\rangle + \langle M_3 x_1\rangle h - m_1 h^2\right)k_1^3 + \mathcal{O}\left(k_1^4\right). \tag{3.83}$$

[4] We use the following convention for the Fourier transform: $\hat{f}(\mathbf{k}) = \mathcal{F}[f](\mathbf{k}) = \iint_{\mathbb{R}^2} e^{2\pi i(k_1x_1+k_2x_2)} f(\mathbf{x})\,dx_1dx_2$.

[5] The crucial observation $\hat{\vec{M}}(\mathbf{0}) = \vec{m}$ openning the doors for the Fourier computations was due to Doug Hardin, Vanderbilt University. This was futher discussed with Eduardo Lima, MIT.

The left-hand side in this expression can be computed using the same idea as before: in the integration range we complement the part known from measurements by another one in which we use the asymptotic development of the field (3.73):

$$\operatorname{Im} \hat{B}_3(k_1, 0, h) = \iint_{\mathbb{R}^2} \sin(2\pi k_1 x_1) B_3(\mathbf{x}, h)\, dx_1 dx_2 = \iint_{D_A} \cdots + \iint_{\mathbb{R}^2 \setminus D_A} \cdots =: \mathcal{U}_1 + \mathcal{E}_1, \tag{3.84}$$

with

$$\mathcal{U}_1 = 2\pi k_1 \iint_{D_A} x_1 B_3(\mathbf{x}, h)\, dx_1 dx_2 - \frac{4\pi^3}{3} k_1^3 \iint_{D_A} x_1^3 B_3(\mathbf{x}, h)\, dx_1 dx_2 + \mathcal{O}\left(k_1^5 A^7,\, k_1^7 A^9, \dots\right), \tag{3.85}$$

$$\mathcal{E}_1 = \frac{3}{4\pi}\left(m_1 h - \langle M_3 x_1 \rangle\right) \iint_{\mathbb{R}^2 \setminus D_A} \frac{x_1 \sin(2\pi k_1 x_1)}{\left(x_1^2 + x_2^2\right)^{5/2}} dx_1 dx_2 + \mathcal{R}_1, \tag{3.86}$$

where next-order terms in $\mathcal{U}_1$ are estimated by means of elementary integration in polar coordinates like

$$k_1^5 \left| \iint_{D_A} x_1^5 B_3(\mathbf{x}, h)\, dx_1 dx_2 \right| \leq \frac{2\pi}{7} \|B_3\|_{L^\infty(D_A)} k_1^5 A^7,$$

while the residue term $\mathcal{R}_1$ will be discussed shortly after.

We are now going to evaluate the leading order term in $\mathcal{E}_1$. Using the integral representation for Bessel function [7, (10.9.1)]

$$J_0(2\pi k_1 r) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(2\pi k_1 r \cos\phi)\, d\phi$$

and its even parity, we write

$$\int_0^{2\pi} \cos\phi \sin(2\pi k_1 r \cos\phi)\, d\phi = 4 \int_0^{\frac{\pi}{2}} \cos\phi \sin(2\pi k_1 r \cos\phi)\, d\phi = -\frac{1}{k_1} \partial_r J_0(2\pi k_1 r) = -2\pi J_0'(2\pi k_1 r).$$

Now since $J_0'(x) = -J_1(x)$ (see (3.109) in Appendix), we can transform

$$\begin{aligned} \iint_{\mathbb{R}^2 \setminus D_A} \frac{x_1 \sin(2\pi k_1 x_1)}{\left(x_1^2 + x_2^2\right)^{5/2}} dx_1 dx_2 &= \int_A^\infty \int_0^{2\pi} \frac{\cos\phi \sin(2\pi k_1 r \cos\phi)}{r^3} d\phi dr = 2\pi \int_A^\infty \frac{J_1(2\pi k_1 r)}{r^3} dr \\ &= 8\pi^3 k_1^2 \operatorname{sgn} k_1 \int_{2\pi|k_1|A}^\infty \frac{J_1(x)}{x^3} dx, \end{aligned}$$

and hence

$$\mathcal{E}_1 = 6\pi^2 \left(m_1 h - \langle M_3 x_1 \rangle\right) k_1 |k_1| \int_{2\pi|k_1|A}^\infty \frac{J_1(x)}{x^3} dx + \mathcal{O}\left(\frac{1}{A^4}\right).$$

Computing the last integral in terms of Bessel and Struve functions (see Appendix for the detailed computation) and using their expansions for small argument [7, (10.2.2), (11.2.1)]

$$J_0(x) = 1 - \frac{1}{4}x^2 + \mathcal{O}\left(x^4\right), \quad J_1(x) = \frac{1}{2}x - \frac{1}{16}x^3 + \mathcal{O}\left(x^5\right), \tag{3.87}$$

$$H_0(x) = \frac{2}{\pi}x + \mathcal{O}\left(x^3\right), \quad H_1(x) = \frac{2}{3\pi}x^2 + \mathcal{O}\left(x^4\right), \tag{3.88}$$

we obtain

$$\int_{2\pi|k_1|A}^{\infty} \frac{J_1(x)}{x^3}dx = \frac{1}{4\pi|k_1|A} - \frac{1}{3} + \frac{\pi}{8}|k_1|A + \mathcal{O}\left(k_1^3A^3\right),$$

and thus

$$\mathcal{E}_1 = 6\pi^2\left(m_1h - \langle M_3x_1\rangle\right)\left(\frac{k_1}{4\pi A} - \frac{1}{3}k_1|k_1| + \frac{\pi}{8}Ak_1|k_1|^2\right) + \mathcal{O}\left(k_1^5A^3\right) + \mathcal{R}_1.$$

We estimate the residue term $\mathcal{R}_1$ by the contribution of the next-order term in asymptotic expansion (3.73) which is, generally speaking, not rigorous but serves the purpose here. Rewritting the corresponding integral in terms of Bessel function and performing iterative integration by parts followed by use of (3.87), we deduce the proportionality

$$\begin{aligned}
\mathcal{R}_1 &\propto \iint_{\mathbb{R}^2\setminus D_A} \frac{x_1\sin(2\pi k_1x_1)}{\left(x_1^2+x_2^2\right)^{7/2}}dx_1dx_2 = 2\pi\int_A^{\infty}\frac{J_1(2\pi k_1 r)}{r^5}dr \\
&= \frac{\pi}{2A^4}J_1(2\pi k_1A) + \frac{\pi^2k_1}{3A^3}J_1'(2\pi k_1A) + \frac{2\pi^3}{3}k_1^2\int_A^{\infty}\frac{J_1''(2\pi k_1r)}{r^3}dr \\
&= \mathcal{O}\left(\frac{k_1}{A^3}, \frac{k_1^3}{A}, k_1^5A, \ldots\right).
\end{aligned}$$

Finally, we plug (3.85)-(3.86) into (3.84) and equate the result to (3.83) at different powers of k_1. We have, at k_1:

$$2\pi\iint_{D_A} x_1B_3(\mathbf{x},h)\,dx_1dx_2 + \frac{3\pi}{2A}\left(m_1h - \langle M_3x_1\rangle\right) + \mathcal{O}\left(\frac{1}{A^3}\right) = \pi m_1, \tag{3.89}$$

which exactly coincides with the already obtained result (3.75).

Factoring out $k_1|k_1|$, we arrive at an identity which does not yield any information, while at order k_1^3 we have

$$-\frac{4\pi^3}{3}\iint_{D_A} x_1^3B_3(\mathbf{x},h)\,dx_1dx_2 + \frac{3\pi^3}{4}A\left(m_1h - \langle M_3x_1\rangle\right) + \mathcal{O}\left(\frac{1}{A}\right) = -2\pi^3\left(-m_1h^2 + \left\langle M_1x_1^2\right\rangle + \langle M_3x_1\rangle h\right).$$

Combining this with (3.89), we can eliminate $(m_1h - \langle M_3x_1\rangle)$ term. This leads to

$$m_1 = 2\iint_{D_A}\left(1 + \frac{4x_1^2}{3A^2}\right)x_1B_3(\mathbf{x},h)\,dx_1dx_2 + \mathcal{O}\left(\frac{1}{A^2}\right), \tag{3.90}$$

which is an improved version of (3.75) in terms of order of approximation without additional assumptions on magnetization distribution.

Similarly,

$$m_2 = 2\iint_{D_A}\left(1 + \frac{4x_2^2}{3A^2}\right)x_2B_3(\mathbf{x},h)\,dx_1dx_2 + \mathcal{O}\left(\frac{1}{A^2}\right). \tag{3.91}$$

To derive more relations involving moments, we take real parts of both sides of (3.82). Analogously to (3.83), we could have restricted ourselves by setting either $k_1 = 0$ or $k_2 = 0$, but instead, wishing to extract more information, we are going to show another approach of asymptotic estimation of integrals that does not involve Bessel functions. We outline the procedure here without giving much details aiming merely at pointing out a possibility of systematic computations. We also note that the same method could certainly be applied when working with (3.83).

Using polar coordinates in the Fourier domain $k_1 = |\mathbf{k}| \cos\theta$, $k_2 = |\mathbf{k}| \sin\theta$, we compute

$$\begin{aligned}
\operatorname{Re} \hat{B}_3(\mathbf{k},h) \;=\;& \pi|\mathbf{k}|\, m_3 - \pi^2|\mathbf{k}|^2\left(2m_3h + \langle M_3x_1\rangle + \langle M_3x_2\rangle\right) - \pi^2|\mathbf{k}|^2\sin 2\theta\left(\langle M_1x_2\rangle + \langle M_2x_1\rangle\right) \\
& -\pi^2|\mathbf{k}|^2\cos 2\theta\left(\langle M_1x_1\rangle - \langle M_2x_2\rangle\right) + \pi^3|\mathbf{k}|^3\left(2m_3h^2 + 2h\left[\langle M_1x_1\rangle + \langle M_2x_2\rangle\right] - \left\langle M_3x_1^2\right\rangle - \left\langle M_3x_2^2\right\rangle\right) \\
& -2\pi^3|\mathbf{k}|^3\sin 2\theta\left(\langle M_3x_1x_2\rangle - h\left[\langle M_1x_2\rangle + \langle M_2x_1\rangle\right]\right) \\
& -\pi^3|\mathbf{k}|^3\cos 2\theta\left(\left\langle M_3x_1^2\right\rangle - \left\langle M_3x_2^2\right\rangle - 2h\left[\langle M_1x_1\rangle - \langle M_2x_2\rangle\right]\right) + \mathcal{O}\left(|\mathbf{k}|^4\right). \qquad (3.92)
\end{aligned}$$

On the other hand, denoting

$$g(\mathbf{k},\phi) := 2\pi|\mathbf{k}|\cos(\theta-\phi), \qquad (3.93)$$

we write

$$\begin{aligned}
\operatorname{Re} \hat{B}_3(\mathbf{k},h) &= \iint_{\mathbb{R}^2}\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)B_3(\mathbf{x},h)\,dx_1dx_2 = \iint_{\mathbb{R}^2}\cos\left[rg(\mathbf{k},\phi)\right]B_3(r\cos\phi, r\sin\phi, h)\,rdrd\phi \\
&= \iint_{D_A}\cdots + \iint_{\mathbb{R}^2\setminus D_A}\cdots =: \mathcal{U}_0 + \mathcal{E}_0. \qquad (3.94)
\end{aligned}$$

Employing (3.73) in $\mathcal{E}_0$ and noticing that, by symmetry,

$$\iint_{\mathbb{R}^2\setminus D_A}\frac{x_1\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)}{\left(x_1^2+x_2^2\right)^{5/2}}dx_1dx_2 = \iint_{\mathbb{R}^2\setminus D_A}\frac{x_2\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)}{\left(x_1^2+x_2^2\right)^{5/2}}dx_1dx_2 = 0,$$

we get

$$\mathcal{E}_0 \simeq -\frac{m_3}{4\pi}\mathcal{I}_2 + \frac{3a_0}{8\pi}\mathcal{I}_4 + \frac{15a_1}{8\pi}\mathcal{I}_4^{(1)} + \frac{15a_2}{8\pi}\mathcal{I}_4^{(2)} + \frac{15a_3}{4\pi}\mathcal{I}_4^{(12)}, \qquad (3.95)$$

where

$$a_0 := -2h\left(\langle M_1x_1\rangle + \langle M_2x_2\rangle\right) + 3m_3h^2 + \left\langle M_3x_1^2\right\rangle + \left\langle M_3x_2^2\right\rangle, \qquad (3.96)$$

$$a_1 := 2h\langle M_1x_1\rangle - \left\langle M_3x_1^2\right\rangle, \quad a_2 := 2h\langle M_2x_2\rangle - \left\langle M_3x_2^2\right\rangle, \qquad (3.97)$$

$$a_3 := h\left(\langle M_1x_2\rangle + \langle M_2x_1\rangle\right) - \langle M_3x_1x_2\rangle, \qquad (3.98)$$

$$\mathcal{I}_2 := \iint_{\mathbb{R}^2\setminus D_A}\frac{\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)}{\left(x_1^2+x_2^2\right)^{3/2}}dx_1dx_2, \quad \mathcal{I}_4 := \iint_{\mathbb{R}^2\setminus D_A}\frac{\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)}{\left(x_1^2+x_2^2\right)^{5/2}}dx_1dx_2, \qquad (3.99)$$

$$\mathcal{I}_4^{(1)} := \iint_{\mathbb{R}^2\setminus D_A}\frac{x_1^2\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)}{\left(x_1^2+x_2^2\right)^{7/2}}dx_1dx_2, \quad \mathcal{I}_4^{(2)} := \iint_{\mathbb{R}^2\setminus D_A}\frac{x_2^2\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)}{\left(x_1^2+x_2^2\right)^{7/2}}dx_1dx_2, \qquad (3.100)$$

$$\mathcal{I}_4^{(12)} := \iint_{\mathbb{R}^2\setminus D_A}\frac{x_1x_2\cos\left(2\pi\left(k_1x_1+k_2x_2\right)\right)}{\left(x_1^2+x_2^2\right)^{7/2}}dx_1dx_2. \qquad (3.101)$$

The key idea of asymptotic expansion here is based on a possibility to express (3.99)-(3.101) in terms of the sine integral special function $\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t} dt$ which admits the following series expansion [7, (6.6.5)]

$$\mathrm{Si}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!\,(2n+1)}. \tag{3.102}$$

Indeed, using the definition (3.93), we integrate by parts to arrive at

$$\mathcal{I}_2 = \int_0^{2\pi} \int_A^{\infty} \frac{\cos[rg(\mathbf{k},\phi)]}{r^2} dr d\phi = \int_0^{2\pi} \left(\frac{1}{A} \cos[Ag(\mathbf{k},\phi)] - \frac{\pi}{2} |g(\mathbf{k},\phi)| + g(k,\phi)\,\mathrm{Si}[Ag(\mathbf{k},\phi)] \right) d\phi.$$

We note that

$$\int_0^{2\pi} |g(\mathbf{k},\phi)|\, d\phi = 2\pi |\mathbf{k}| \int_0^{2\pi} |\cos\phi|\, d\phi = 8\pi |\mathbf{k}|,$$
$$\int_0^{2\pi} \cos^{2n}(\theta-\phi)\, d\phi = \int_0^{2\pi} \cos^{2n}\phi d\phi = \frac{2\pi}{4^n} \frac{(2n)!}{(n!)^2}, \quad n \in \mathbb{N}_0,$$

where the second integral is computed by multi-angle identities obtained from binomial formula and Euler trigonometric formula or using Chebyshev polynomials, while the remarkable independence of θ is easy to see by vanishing of the integral of the derivative.

Employing these integral computations, (3.102) and Taylor expansion for cosine, we obtain

$$\begin{aligned} \mathcal{I}_2 &= \frac{2\pi}{A} - 4\pi^2 |\mathbf{k}| - 2\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(n!)^2 (2n-1)} A^{2n-1} |\mathbf{k}|^{2n} \\ &= \frac{2\pi}{A} - 4\pi^2 |\mathbf{k}| + 2\pi^3 A |\mathbf{k}|^2 + \mathcal{O}\left(A^3 |\mathbf{k}|^4,\, A^5 |\mathbf{k}|^6, \ldots\right). \end{aligned}$$

Similarly, we compute

$$\mathcal{I}_4 = \int_0^{2\pi} \int_A^{\infty} \frac{\cos[rg(\mathbf{k},\phi)]}{r^4} dr d\phi = \frac{1}{3} \int_0^{2\pi} \left(\frac{\cos[Ag(\mathbf{k},\phi)]}{A^3} - \frac{1}{2A^2} \sin[Ag(\mathbf{k},\phi)] - \frac{g(\mathbf{k},\phi)}{2} \int_A^{\infty} \frac{\cos[rg(\mathbf{k},\phi)]}{r^2} dr \right) d\phi,$$

where the last integral term can be treated as above by means of reduction to the sine integral function.

Performing Taylor expansions and taking into account that

$$\int_0^{2\pi} |\cos(\theta-\phi)|^3\, d\phi = \int_0^{2\pi} |\cos\phi|^3\, d\phi = \frac{8}{3},$$

we get

$$\begin{aligned} \mathcal{I}_4 &= \frac{2\pi}{3A^3} - \frac{2\pi^3}{A} |\mathbf{k}|^2 + \frac{16\pi^4}{9} |\mathbf{k}|^3 + \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^n [2(n+1)]! \pi^{2n+3}}{(2n+1)!\,[(n+1)!]^2} \left(\frac{2n+1}{2n-1} + \frac{n}{n+1} \right) A^{2n-1} |\mathbf{k}|^{2(n+1)} \\ &= \frac{2\pi}{3A^3} - \frac{2\pi^3}{A} |\mathbf{k}|^2 + \frac{16\pi^4}{9} |\mathbf{k}|^3 + \mathcal{O}\left(A |\mathbf{k}|^4,\, A^3 |\mathbf{k}|^6, \ldots\right). \end{aligned}$$

To estimate $\mathcal{I}_4^{(1)}$, $\mathcal{I}_4^{(2)}$, we also compute the quantity

$$\begin{aligned}
\mathcal{I}_4^{(0)} &:= \int_0^{2\pi}\int_A^{\infty} \cos 2\phi \frac{\cos\left[rg\left(\mathbf{k},\phi\right)\right]}{r^4} dr d\phi \\
&= -\frac{\pi^3}{A}\left|\mathbf{k}\right|^2 \cos 2\theta + \frac{16}{15}\pi^4 \left|\mathbf{k}\right|^3 \cos 2\theta \\
&\quad + \frac{\cos 2\theta}{3}\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+3}}{(2n+1)!}\left(\frac{[2(n+2)]!}{[(n+2)!]^2} - \frac{[2(n+1)]!}{[(n+1)!]^2}\right)\left(\frac{2n+1}{2n-1} + \frac{n}{n+1}\right) A^{2n-1}\left|\mathbf{k}\right|^{2(n+1)} \\
&= -\frac{\pi^3}{A}\left|\mathbf{k}\right|^2 \cos 2\theta + \frac{16}{15}\pi^4 \left|\mathbf{k}\right|^3 \cos 2\theta + \mathcal{O}\left(A\left|\mathbf{k}\right|^4,\ A^3\left|\mathbf{k}\right|^6, \dots\right).
\end{aligned}$$

It now follows that

$$\mathcal{I}_4^{(1)} = \frac{1}{2}\left(\mathcal{I}_4 + \mathcal{I}_4^{(0)}\right) = \frac{\pi}{3A^3} - \frac{\pi^3}{A}\left|\mathbf{k}\right|^2 - \frac{\pi^3}{2A}\left|\mathbf{k}\right|^2 \cos 2\theta + \frac{8}{9}\pi^4\left|\mathbf{k}\right|^3 + \frac{8}{15}\pi^4\left|\mathbf{k}\right|^3 \cos 2\theta + \mathcal{O}\left(A\left|\mathbf{k}\right|^4,\ A^3\left|\mathbf{k}\right|^6, \dots\right),$$

$$\mathcal{I}_4^{(2)} = \frac{1}{2}\left(\mathcal{I}_4 - \mathcal{I}_4^{(0)}\right) = \frac{\pi}{3A^3} - \frac{\pi^3}{A}\left|\mathbf{k}\right|^2 + \frac{\pi^3}{2A}\left|\mathbf{k}\right|^2 \cos 2\theta + \frac{8}{9}\pi^4\left|\mathbf{k}\right|^3 - \frac{8}{15}\pi^4\left|\mathbf{k}\right|^3 \cos 2\theta + \mathcal{O}\left(A\left|\mathbf{k}\right|^4,\ A^3\left|\mathbf{k}\right|^6, \dots\right).$$

Finally, we calculate

$$\begin{aligned}
\mathcal{I}_4^{(12)} &= \frac{1}{2}\int_0^{2\pi}\int_A^{\infty} \sin 2\phi \frac{\cos\left[rg\left(\mathbf{k},\phi\right)\right]}{r^4} dr d\phi \\
&= -\frac{\pi^3}{3A}\left|\mathbf{k}\right|^2 \sin 2\theta + \frac{8\pi^4}{15}\left|\mathbf{k}\right|^3 \sin 2\theta \\
&\quad + \frac{\sin 2\theta}{6}\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+3}}{(2n+1)!}\left(\frac{[2(n+2)]!}{[(n+2)!]^2} - \frac{[2(n+1)]!}{[(n+1)!]^2}\right)\left(\frac{2n+1}{2n-1} + \frac{n}{n+1}\right) A^{2n-1}\left|\mathbf{k}\right|^{2(n+1)} \\
&= -\frac{\pi^3}{3A}\left|\mathbf{k}\right|^2 \sin 2\theta + \frac{8\pi^4}{15}\left|\mathbf{k}\right|^3 \sin 2\theta + \mathcal{O}\left(A\left|\mathbf{k}\right|^4,\ A^3\left|\mathbf{k}\right|^6, \dots\right).
\end{aligned}$$

The computed quantities (3.99)-(3.101) furnish $\mathcal{E}_0$. Combined with

$$\begin{aligned}
\mathcal{U}_0 &= \iint_{D_A} \cos\left(2\pi\left|\mathbf{k}\right|\left(x_1\cos\theta + x_2\sin\theta\right)\right) B_3\left(\mathbf{x},h\right) dx_1 dx_2 \\
&= \iint_{D_A} B_3\left(\mathbf{x},h\right) dx_1 dx_2 - \pi^2\left|\mathbf{k}\right|^2 \iint_{D_A}\left(x_1^2 + x_2^2\right) B_3\left(\mathbf{x},h\right) dx_1 dx_2 - \pi^2\left|\mathbf{k}\right|^2 \cos 2\theta \iint_{D_A}\left(x_1^2 - x_2^2\right) B_3\left(\mathbf{x},h\right) dx_1 dx_2 \\
&\quad -2\pi^2\left|\mathbf{k}\right|^2 \sin 2\theta \iint_{D_A} x_1 x_2 B_3\left(\mathbf{x},h\right) dx_1 dx_2 + \mathcal{O}\left(\left|\mathbf{k}\right|^4 A^6,\ \left|\mathbf{k}\right|^6 A^8, \dots\right),
\end{aligned}$$

we plug this in (3.94) and equate to (3.92) matching expressions at different powers of $|\mathbf{k}|$ and presence of θ-dependent factors. Matching at neighbourhoods $|\mathbf{k}|$, $|\mathbf{k}|^3$, $|\mathbf{k}|^3\cos 2\theta$, $|\mathbf{k}|^3\sin 2\theta$ yields identities whereas the neighbourhoods $|\mathbf{k}|^0$, $|\mathbf{k}|^2$, $|\mathbf{k}|^2\cos 2\theta$, $|\mathbf{k}|^2\sin 2\theta$ produce the following, respectively

$$\iint_{D_A} B_3\left(\mathbf{x},h\right) dx_1 dx_2 - \frac{m_3}{2A} + \frac{2a_0 + 5\left(a_1 + a_2\right)}{8A^3} + \mathcal{O}\left(\frac{1}{A^5}\right) = 0,$$

$$-\pi^2\iint_{D_A}\left(x_1^2 + x_2^2\right) B_3\left(\mathbf{x},h\right) dx_1 dx_2 - \frac{m_3\pi^2 A}{2} - \frac{3\pi^2\left[2a_0 + 5\left(a_1 + a_2\right)\right]}{2A} + \mathcal{O}\left(\frac{1}{A^3}\right) = -\pi^2 q_0,$$

$$-\pi^2\iint_{D_A}\left(x_1^2 - x_2^2\right) B_3\left(\mathbf{x},h\right) dx_1 dx_2 - \frac{15\pi^2}{4A}\left(a_1 - a_2\right) + \mathcal{O}\left(\frac{1}{A^3}\right) = -\pi^2\left(\langle M_1 x_1\rangle - \langle M_2 x_2\rangle\right),$$

$$-2\pi^2 \iint_{D_A} x_1 x_2 B_3(\mathbf{x}, h)\, dx_1 dx_2 - \frac{5\pi^2}{4A} a_3 + \mathcal{O}\left(\frac{1}{A^3}\right) = -\pi^2 \left(\langle M_1 x_2 \rangle + \langle M_2 x_1 \rangle\right),$$

where q_0 is as in (3.56).

The first of these expressions gives the same as in (3.78) that was obtained by other means while its combination with the second one produces an estimate for the quantity q_0 arisen also in the Kelvin transform method (see (3.62)). These equations, along with others that can be obtained by this approach systematically (note that the algorithmic computations of expansions demonstrated above persists when (3.73) is expanded further), produce more subtle information involving quantities (3.96)-(3.98). Such information becomes especially valuable, for instance, when one makes a smallness assumption on higher-order algebraic moments of magnetization (see discussion around (3.76)-(3.77)) since this helps to improve accuracy by forming combinations eliminating higher-order terms in $1/A$.

3.7 Numerical illustrations

First, we are going to illustrate net moment reconstruction using limiting formulas obtained for the case of completely available data, namely those given by (3.35)-(3.36), (3.50)-(3.51), (3.59), (3.62) and (3.72).

We consider the synthethic example with $N = 4$ dipoles with the moments $\vec{m}^{(1)} = (0.9, 0.7, 0.2)^T$, $\vec{m}^{(2)} = (0.5, 0.9, 0.1)^T$, $\vec{m}^{(3)} = (-0.6, 0.4, 0.5)^T$, $\vec{m}^{(4)} = (-0.2, 0.4, 0.3)^T$ placed at the locations $\mathbf{x}^{(1)} = (0.7, 0.6)^T$, $\mathbf{x}^{(2)} = (0.0, 0.0)^T$, $\mathbf{x}^{(3)} = (0.8, -1.1)^T$, $\mathbf{x}^{(4)} = (-0.8, 1.1)^T$ in the plane $x_3 = 0$. By superposition of dipolar fields, this produces the following potential and field at height $h = 5$:

$$\Phi(\mathbf{x}, h) = \frac{1}{4\pi} \sum_{l=1}^{N} \frac{m_1^{(l)}\left(x_1 - x_1^{(l)}\right) + m_2^{(l)}\left(x_2 - x_2^{(l)}\right) + m_3^{(l)} h}{\left[\left(x_1 - x_1^{(l)}\right)^2 + \left(x_2 - x_2^{(l)}\right)^2 + h^2\right]^{3/2}},$$

$$B_3(\mathbf{x}, h) = \frac{1}{4\pi} \sum_{l=1}^{N} \frac{3h\left[m_1^{(l)}\left(x_1 - x_1^{(l)}\right) + m_2^{(l)}\left(x_2 - x_2^{(l)}\right)\right] + m_3^{(l)}\left(2h^2 - \left(x_1 - x_1^{(l)}\right)^2 - \left(x_2 - x_2^{(l)}\right)^2\right)}{\left[\left(x_1 - x_1^{(l)}\right)^2 + \left(x_2 - x_2^{(l)}\right)^2 + h^2\right]^{5/2}}.$$

Figures 3.7.1-3.7.3 show the estimation of net moment components m_1, m_2 and the combination q_0, respectively, from the expressions in terms of potential and field as functions of the limiting variable R_0. To mimic the knowledge of both potential and field in the entire plane, we evaluate these expressions over a disk of radius $A = 10000$ centered at the origin.

On Figure 3.7.4, the estimate of net moment component m_3 is plotted against the limiting variable ϱ which in this case has a simple geometric meaning - the radius of the circle from which the field data are taken (see (3.72)).

We then go on to demonstrate formulas (3.78)-(3.80) giving the components of net moment in the case of partial data. Obviously, quality of these formulae depends on the size of the measurement area: the bigger the area, the better the accuracy. To illustrate this graphically, we let the disk radius A vary while keeping other parameters fixed as before. This produces Figures 3.7.5-3.7.6. On Figure 3.7.5, we compare results with higher-order estimates (3.90)-(3.91) furnished by Fourier analysis whereas the obtained earlier formula (3.78) appearing on Figure 3.7.6

is already of second order of accuracy.

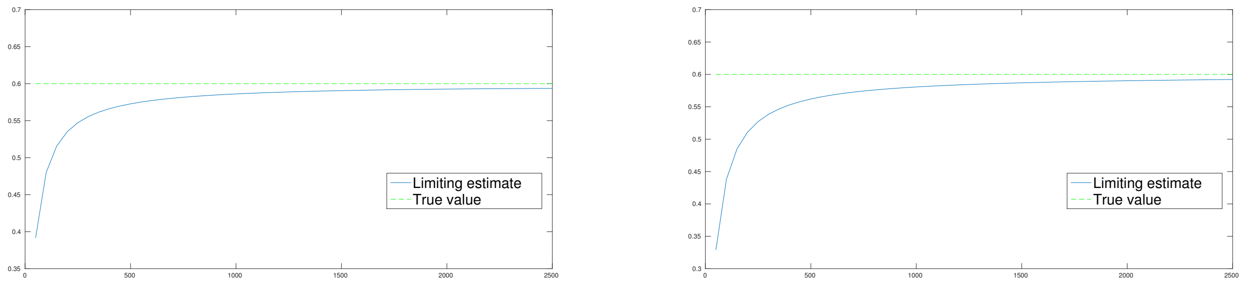

Figure 3.7.1: m_1 estimated from potential and field expressions (3.35), (3.50), respectively. Case of complete data.

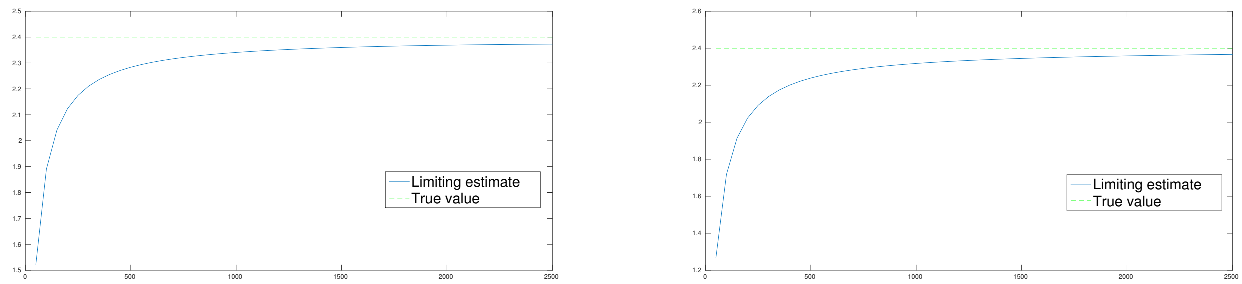

Figure 3.7.2: m_2 estimated from potential and field expressions (3.36), (3.51), respectively. Case of complete data.

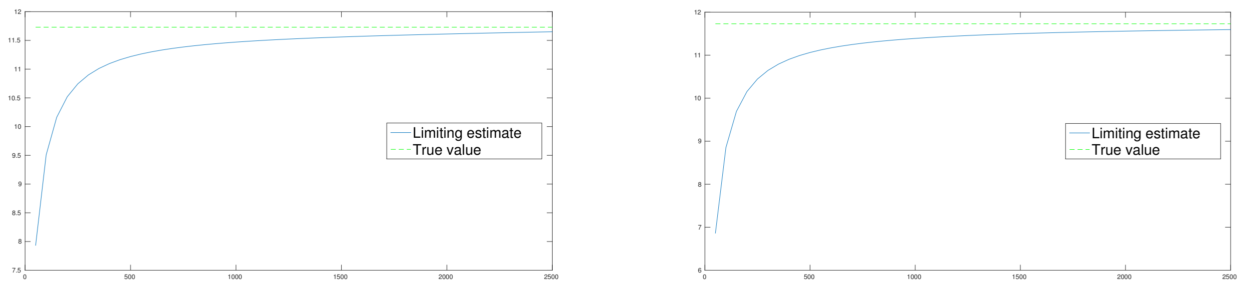

Figure 3.7.3: q_0 estimated from potential and field expressions (3.59), (3.62), respectively. Case of complete data.

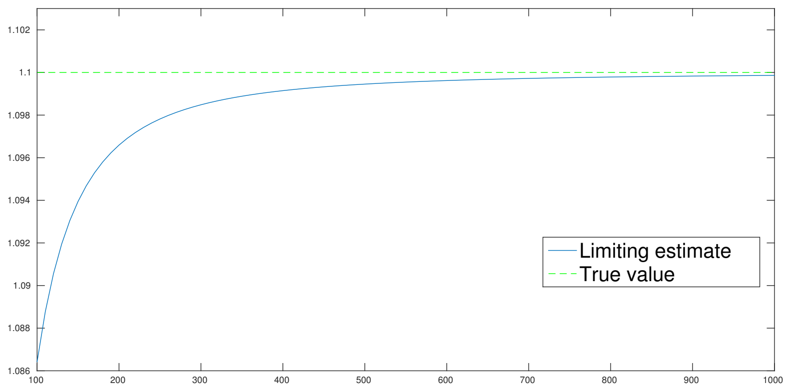

Figure 3.7.4: m_3 estimated from (3.72). Case of complete data.

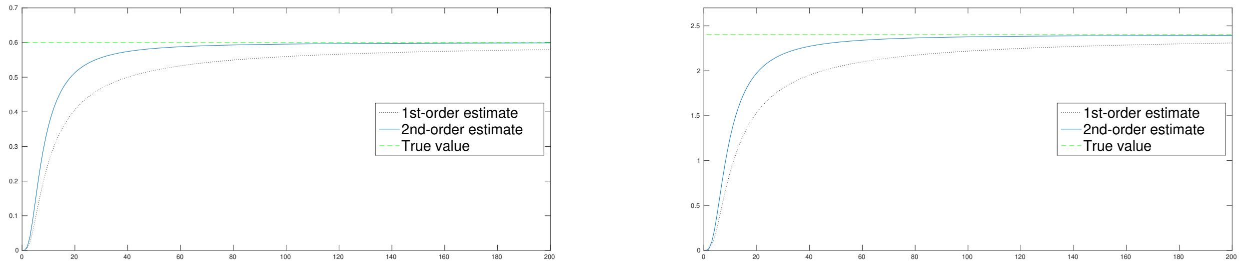

Figure 3.7.5: m_1, m_2 estimated from (3.79)-(3.80) and (3.90)-(3.91). Case of partial data.

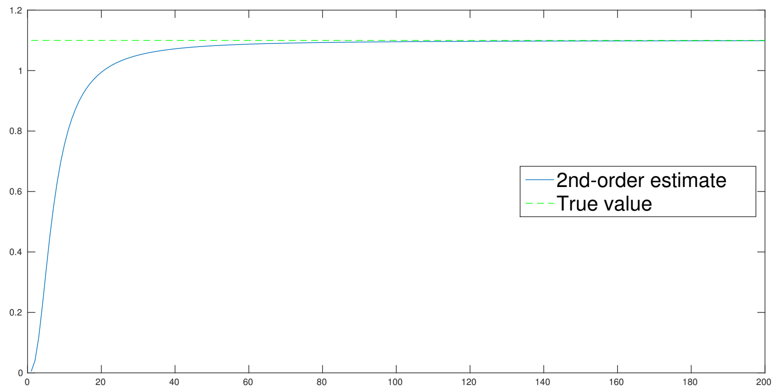

Figure 3.7.6: m_3 estimated from (3.78). Case of partial data.

APPENDIX

Some integral computations

- $$\int \frac{(c_1\omega + c_2)\,\omega^2}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega = (3\alpha c_1 + c_2)\,\mathrm{arcsinh}\frac{\omega-\alpha}{\sqrt{c_0}} + c_1\left[(\omega-\alpha)^2 + c_0\right]^{1/2} - \frac{3\alpha^2 c_1 - c_0 c_1 + 2\alpha c_2}{\left[(\omega-\alpha)^2 + c_0\right]^{1/2}} - \left[(2\alpha c_1 + c_2)\left(c_0 + \alpha^2\right) - \alpha\left(3\alpha^2 c_1 - c_0 c_1 + 2\alpha c_2\right)\right] \frac{\omega-\alpha}{c_0\left[(\omega-\alpha)^2 + c_0\right]^{1/2}}.$$

Proof.

$$\begin{aligned}
\int \frac{(c_1\omega + c_2)\,\omega^2}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega &= \int \frac{c_1\omega + c_2}{\left[(\omega-\alpha)^2 + c_0\right]^{1/2}} d\omega - \int \frac{\left(c_0 + \alpha^2 - 2\alpha\omega\right)(c_1\omega + c_2)}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega \\
&= c_1\left[(\omega-\alpha)^2 + c_0\right]^{1/2} + (3\alpha c_1 + c_2)\int \frac{d\omega}{\left[(\omega-\alpha)^2 + c_0\right]^{1/2}} - \frac{\left(3\alpha^2 c_1 - c_0 c_1 + 2\alpha c_2\right)}{\left[(\omega-\alpha)^2 + c_0\right]^{1/2}} \\
&\quad - \left[(2\alpha c_1 + c_2)\left(c_0 + \alpha^2\right) - \alpha\left(3\alpha^2 c_1 - c_0 c_1 + 2\alpha c_2\right)\right]\int \frac{d\omega}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}}.
\end{aligned}$$

□

In the same fashion, one can compute the next integral

- $$\int \frac{(c_1\omega + c_2)\,\omega}{\left[(\omega-\alpha)^2 + c_0\right]^{3/2}} d\omega = c_1 \mathrm{arcsinh}\frac{\omega-\alpha}{\sqrt{c_0}} - \frac{2\alpha c_1 + c_2}{\left[(\omega-\alpha)^2 + c_0\right]^{1/2}} - \frac{\left(c_0 c_1 - \alpha^2 c_1 - \alpha c_2\right)(\omega-\alpha)}{c_0\left[(\omega-\alpha)^2 + c_0\right]^{1/2}}.$$

- $$\int_A^\infty \frac{1}{r^2\left(r^2 + R_0^2\right)^{5/2}} dr = -\frac{8}{3R_0^6} + \frac{3R_0^4 + 4A^2\left(3R_0^2 + 2A^2\right)}{3R_0^6 A\left(A^2 + R_0^2\right)^{3/2}}, \quad A > 0.$$

Proof. The key element of the calculations is the integral

$$\int \frac{dx}{(\alpha + x)^{1/2}(\beta + x)^{3/2}} = \frac{2}{\beta - \alpha}\left(\frac{\alpha + x}{\beta + x}\right)^{1/2}, \tag{3.103}$$

which can be readily computed by the change of variable $t = \dfrac{\alpha + x}{\beta + x}$.

Differentiating this result with respect to α yields

$$S(\alpha, \beta) := \int \frac{dx}{(\alpha + x)^{3/2}(\beta + x)^{3/2}} = -\frac{2}{(\beta - \alpha)^2}\left[\left(\frac{x + \alpha}{x + \beta}\right)^{1/2} + \left(\frac{x + \beta}{x + \alpha}\right)^{1/2}\right].$$

We carry on to evaluate

$$\int \frac{dx}{x^{5/2}(\beta + x)^{3/2}} = -\frac{2}{3}\frac{\partial S}{\partial \alpha}(0, \beta) = \frac{2}{3}\frac{4x(2x + \beta) - \beta^2}{\beta^3 x^{3/2}(x + \beta)^{1/2}}, \tag{3.104}$$

$$\int \frac{dx}{x^{5/2}(\beta+x)^{5/2}} = \frac{4}{9}\frac{\partial^2 S}{\partial\alpha\partial\beta}(0,\beta) = \frac{2}{3}\frac{16x^3+24\beta x^2+6\beta^2 x-\beta^3}{\beta^4 x^{3/2}(x+\beta)^{3/2}}. \tag{3.105}$$

Now these integrals furnish the desired result by means of the decomposition

$$\int_A^\infty \frac{dr}{r^2\left(r^2+R_0^2\right)^{5/2}} = \frac{1}{2}\int_{A^2}^\infty \frac{dx}{x^{5/2}\left(x+R_0^2\right)^{3/2}} - \frac{R_0^2}{2}\int_{A^2}^\infty \frac{dx}{x^{5/2}\left(x+R_0^2\right)^{5/2}}.$$

□

- $$\int_A^\infty \frac{dr}{r^4\left(r^2+R_0^2\right)^{5/2}} = \frac{16}{3R_0^8} - \frac{1}{3}\frac{16A^6+24R_0^2A^4+6R_0^4A^2-R_0^6}{R_0^8A^3\left(A^2+R_0^2\right)^{3/2}}, \quad A>0.$$

Proof. As before, by the change of variable, we have

$$\int_A^\infty \frac{dr}{r^4\left(r^2+R_0^2\right)^{5/2}} = \frac{1}{2}\int_{A^2}^\infty \frac{dx}{x^{5/2}\left(x+R_0^2\right)^{3/2}},$$

and the result follows by application of (3.105). □

- $$\begin{aligned}\int_C^\infty \frac{J_1(x)}{x^3}dx \quad = \quad & -\frac{1}{3}\left[1-\left(1+\frac{1}{C^2}\right)CJ_0(C)+\left(1-\frac{1}{C^2}\right)J_1(C)\right.\\ & \left.+\frac{\pi C}{2}\left(J_0(C)H_1(C)-J_1(C)H_0(C)\right)\right], \quad C>0.\end{aligned}$$

Proof. In order to evaluate this integral, we will repeatedly employ a few well-known properties of Bessel functions [7, Sect. 10.6]. We recall Bessel differential equation [7, Sect. 10.2]

$$x^2J_n''(x)+xJ_n'(x)+\left(x^2-n^2\right)J_n(x)=0 \quad \Leftrightarrow \quad \frac{1}{x}J_n'(x)=\left(\frac{n^2}{x^2}-1\right)J_n(x)-J_n''(x), \tag{3.106}$$

and we will also heavily rely on the recurrence formulas

$$\frac{1}{x}J_n(x)=\frac{1}{2n}\left(J_{n-1}(x)+J_{n+1}(x)\right), \tag{3.107}$$

$$J_n'(x)=\frac{1}{2}\left(J_{n-1}(x)-J_{n+1}(x)\right), \tag{3.108}$$

implying, in particular, that

$$J_0'(x)=-J_1(x), \tag{3.109}$$

$$J_3(x)=\left(\frac{8}{x^2}-1\right)J_1(x)-\frac{4}{x}J_0(x). \tag{3.110}$$

We start by using (3.107) with $n=1$ to compute the indefinite integral

$$\int\frac{J_1(x)}{x^3}dx=\frac{1}{2}\int\frac{J_0(x)}{x^2}dx+\frac{1}{2}\int\frac{J_2(x)}{x^2}dx. \tag{3.111}$$

The first term on the right we integrate by parts and use the differential equation (3.108) for $J_0(x)$ to obtain

$$\int \frac{J_0(x)}{x^2}dx = -\frac{J_0(x)}{x} + \int \frac{J_0'(x)}{x}dx = -\frac{J_0(x)}{x} - J_0'(x) - \int J_0(x)\,dx. \tag{3.112}$$

Application of the same strategy to the second term in the right-hand side of (3.111) is not immediately beneficial due to the presence of an extra term in the equation (3.108) for $n = 2$, however, it still yields

$$\int \frac{J_2(x)}{x^2}dx = -\frac{J_2(x)}{x} - J_2'(x) - \int J_2(x)\,dx + 4\int \frac{J_2(x)}{x^2}dx$$

$$\Rightarrow \quad \int \frac{J_2(x)}{x^2}dx = \frac{1}{3}\left(\frac{J_2(x)}{x} + J_2'(x) + \int J_2(x)\,dx\right). \tag{3.113}$$

We notice that $\int J_2(x)\,dx$ expresses in terms of $\int J_0(x)\,dx$, another ingredient that we have. Indeed, from the integral representation of Bessel functions

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\,dt, \tag{3.114}$$

we directly get

$$\int J_n(x)\,dx = \frac{1}{\pi}\int_0^\pi \frac{\sin(x\sin t - nt)}{\sin t}dt = \frac{1}{\pi}\int_0^\pi \frac{\cos(nt)\sin(x\sin t)}{\sin t}dt - \frac{1}{\pi}\int_0^\pi \frac{\sin(nt)\cos(x\sin t)}{\sin t}dt,$$

and, in particular,

$$\int J_2(x)\,dx = \frac{1}{\pi}\int_0^\pi \frac{\left(1-2\sin^2 t\right)\sin(x\sin t)}{\sin t}dt - \frac{2}{\pi}\int_0^\pi \frac{\cos t\cos(x\sin t)}{\sin t}dt = \int J_0(x)\,dx - 2J_1(x). \tag{3.115}$$

Now getting back to (3.111), we plug (3.112)-(3.113) and use (3.107)-(3.108) with $n = 2$ and (3.109) to arrive at

$$\int \frac{J_1(x)}{x^3}dx = -\frac{1}{2}\frac{J_0(x)}{x} + \frac{5}{8}J_1(x) - \frac{1}{24}J_3(x) - \frac{1}{2}\int J_0(x)\,dx + \frac{1}{6}\int J_2(x)\,dx.$$

From here, making use of (3.110) and (3.115), we obtain

$$\int \frac{J_1(x)}{x^3}dx = -\frac{1}{3}\left(\frac{J_0(x)}{x} - \left(1 - \frac{1}{x^2}\right)J_1(x) + \frac{1}{2}\int J_0(x)\,dx\right). \tag{3.116}$$

It remains to compute the last integral term on the right which can be done in terms of other special functions. To this effect, we use [7, (10.22.2)] with $\nu = 0$

$$\int J_0(x)\,dx = \frac{\pi}{2}x\left(J_0(x)\,H_{-1}(x) - J_{-1}(x)\,H_0(x)\right), \tag{3.117}$$

where Struve functions H_n can be written in terms of Euler gamma functions as [7, (11.2.1)]

$$H_n(x) = \left(\frac{x}{2}\right)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2k},$$

and they also admit an integral representation [7, (11.5.1)]

$$H_n(x) = \frac{n!}{(2n)!}(2x)^n \frac{2}{\pi}\int_0^{\pi/2} \cos^{2n} t \sin(x \sin t)\, dt, \quad n \in \mathbb{N}_0. \tag{3.118}$$

In particular, using also a recurrence formula [7, (11.4.23)], we have

$$H_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin t)\, dt, \tag{3.119}$$

$$H_{-1}(x) = \frac{2}{\pi} - H_1(x) = \frac{2}{\pi}\left(1 - x\int_0^{\pi/2} \cos^2 t \sin(x\sin t)\, dt\right). \tag{3.120}$$

Computing the definite integral require knowledge of the limiting behavior at infinity. To deduce it, we need representation of Struve functions of first kind in terms of Neumann functions and Struve functions of second kind, $Y_n(x)$ and $K_n(x)$, respectively, and their asymptotics as $x \to \infty$ [7, (11.2.5), (11.6.1)]

$$H_n(x) = K_n(x) + Y_n(x),$$

$$K_n(x) = \frac{2^{n+1} n!}{\sqrt{\pi}\,(2n)!} x^{n-1} + \mathcal{O}\left(x^{n-3}\right),$$

$$Y_n(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{x}\right).$$

Recalling also asymptotics of Bessel functions of the first kind for large values of x

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{x}\right),$$

we can now evaluate

$$\begin{aligned}
\lim_{x\to+\infty} \int \frac{J_1(x)}{x^3} dx &= -\frac{\pi}{6} \lim_{x\to+\infty} x\left(J_0(x) H_{-1}(x) - J_{-1}(x) H_0(x)\right) \\
&= -\frac{\pi}{6} \lim_{x\to+\infty} x\left[\frac{2}{\pi x}\cos^2\left(x - \frac{\pi}{4}\right) + \frac{2}{\pi x}\sin^2\left(x - \frac{\pi}{4}\right)\right] \\
&= -\frac{1}{3}.
\end{aligned}$$

Finally, employing (3.117) and (3.120), we conclude the result from (3.116). □

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Concluding remarks

In Part 1, we have considered an ill-posed overdetermined problem for Laplace equation on the planar domain (disk). We proposed a non-iterative regularization scheme based on a newly obtained method of estimation of approximation quality. Another interesting feature of the method is in that it allows to prescribe localized values at points in the interior of the domain. Since the formulated problem is rather general it would be desirable to find a particular application where advantage of the method can be taken to the full extent. One of such possibilities is to improve interpretation of boundary measurements, viewing some of the data points as internal pointwise constraints rather than boundary values. With respect to that possibility, there are number of issues one may further want to look into. For example, it would be interesting to see how the choice of measurement positions, where pointwise data are given, affects the solution. How does an increase of number of points boost the approximation rate and lower the discrepancy growth? With the same density, are the results better when points are located closer to the boundary, when they are spread out evenly in the disk or clustered or put along a curve? Physically, if positions of sensors do not lie along a regular line, does it worth singling out some far out points to be excluded from interpolation of boundary data functions in order to be interpreted as internal constraints? More numerical experiments with the already developed software are needed to gain some intuition for answering such questions.

Part 2 was concerned with spectral structure of the truncated Poisson operator with main focus on developing a method for asymptotic constructions of eigenfunctions for two regimes: when a geometric parameter $\beta = h/a$ is large and when it is small. In the former case, the integral equation was approximated by another one which is reducible to a differential equation. Even though we provided some explanations regarding our expectations about the approximation quality, the error term in this approximation has yet to be studied rigorously. In the case, when β is small, by a chain of transformations, we reduced the integral equation on an interval to the integro-differential problem on a half-line which we were able to solve approximately and then construct continuation of the solution back to the interval of interest. From the half-line problem, we also deduce an approximate relation between the derivative and the value of the function at the endpoints. By the continuity through the endpoints, the same boundary condition can be imposed on the solution extended back to the interval. Consequently, imposing this boundary condition (or, alternatively, matching interior and exterior solutions), we obtained, separately for the cases of even and odd parity, approximations to eigenfunctions and approximate characteristic equations whose solutions are precisely eigenvalues. Expressions for even and odd eigenfunctions have, respectively, cosine and sine terms with frequencies that are logarithms of eigenvalues. We further show numerically that the faithful (non-asymptotic) solutions are very well approximated by sines and cosines already for significant number of eigenfunctions even when the asymptotic parameter is not very small. However, the deviation from sines and cosines families becomes visible for eigenfunctions of higher index, and yet this deviation is localized near the interval endpoints. The similar feauture is observed when comparing numerical results with asymptotic solutions for another range of asymptotic parameter ($\beta \ll 1$) obtained by different analysis allowing to deduce that solutions are close to some standard set of special functions, namely, scaled versions of prolate spheroidal wave functions. Important outcome of such an observation is that even in the asymptotical case (either $\beta \gg 1$ or $\beta \ll 1$), the integral equation cannot be reduced to another one falling in a conventional solvable class whose solutions are purely trigonometric (though with frequencies that cannot be explicitly found).

Another asymptotic strategy is to use the discussed operator approximation and then solve approximately the Prandtl equation. Along a similar line of reasoning, the ongoing work is also dedicated to a delicate method based on reduction to a differential Riemann problem.

Pointwise constraints that Fourier transform of the solution must satisfy have been formulated, however, further advance is yet to be made in this direction. Perhaps some sampling properties of spaces might be used or, alternatively, another functional equation can be constructed by combining infinite number of such discrete relations.

A tempting aspect of the discussed Poisson operator approximation leading to a hypersingular Prandtl integral equation (which has also been a subject of numerous physical works for over five decades) is to construct its approximate solutions employing its connection with our Poisson integral equation that could be solved by other means.

Part 3 of the thesis was different in that it is very closely related to a particular physical application, namely, reconstruction of the net moment vector of magnetization of a finite size sample. Even though the developed methods also apply for rather general problems, we formulated our results with focus on a specific set-up and using the terminology of a concrete geophysical problem of paleomagnetism. The net moment problem was constructively solved exactly by a limiting formula in case of fully available data and asymptotically, in case of partial data, using two newly developed technics based on Fourier and Kelvin transformations and asymptotic continuation of the data. We discovered that a certain set of scalar quantities (algebraic moments of magnetization) can be extracted from the data using both methods. With the Kelvin transformation method, this set is generated by integration against different spherical harmonics while in the Fourier domain - by means of asymptotic matching in different neighborhoods of the origin depending also on a direction along which it is approached. The net moment formulas were first obtained with help of Kelvin transformation whereas Fourier asymptotic matching method developed after seems to be more illuminating and, in particular, it was successfully employed to generate second-order asymptotic formulas for the tangential components of the net moment. The natural question is that can we, in a same resultative way, combine the other scalar quantities extractable from data? Their derivation is tedious yet systematic and hence can be performed with help of computer algebraic systems. It has been discussed that this goal can be easily achieved under additional assumptions on magnetization distribution (smallness of algebraic moments of some higher order), but the general question still remains.

Regarding the obtained asymptotic net moment formulas, we would like to point out another observation demonstrating the following qualitative result: if the measurements are available with perfect resolution even though being incomplete (available only within the disk of a fixed radius), the asymptotic net moment formulas can be improved to arbitrary high precision. This can be seen, first, by noticing that the asymptotic expansion is given in terms of inverse powers of the disk radius, and hence, by repeated differentiation of the formulas with respect to the radius, it is possible to form linear combinations consequently eliminating lower-order error terms. The obtained higher-order formulas will, therefore, be given in terms of integrals and radial derivatives of the field on the boundary of the disk that can be numerically approximated by "backward differences" in terms of values of the field inside the disk.

Practical version of this would be combining few measurements for disks of different radii. Linear combination of each two yields reduction of the error term by one order of magnitude. In this situation, a trade-off has to be found between sacrificing data points when cropping the measurement area significantly (while still staying within asymptotic regime) on one hand, and reliability of the data from a small area of thin ring on the other hand.

Finally, we want to stress that the choice of measurement area as a disk was not crucial. It is possible to obtain similar formulas for net moments (which would differ only by numerical constants depending on geometry of the area) for an area of any shape, though to obtain explicit integral estimates, one would, certainly, prefer the areas of symmetric shapes such as disk or square.